

# A LOCAL UNIQUENESS RESULT FOR AN INVERSE PROBLEM TO THE SYSTEM MODELLING NONHOMOGENEOUS ASYMMETRIC FLUIDS

ANÍBAL CORONEL<sup>†</sup> AND MARKO ROJAS-MEDAR<sup>†</sup>

**ABSTRACT.** In this paper, we prove the local uniqueness of an inverse problem arising in the nonstationary flow of a nonhomogeneous incompressible asymmetric fluid in a bounded domain with smooth boundary. The direct problem is an initial-boundary value problem for a system for the velocity field, the angular velocity of rotation of the fluid particles, the mass density and the pressure distribution. The inverse problem consists in the external force recover assuming an integral measurements on the boundary. We characterize the inverse problem solutions using an operator equation of a second kind, which is deduced from the application of the Helmholtz decomposition. We introduce several estimates which implies the hypothesis of the Tikhonov fixed point theorem.

## 1. INTRODUCTION

The fluids with density-dependent and non-symmetric behavior of the stress tensor belong to a widely class of fluids which are relevant in many industrial applications and in several areas of science. Concerning to the industrial and laboratory applications, this kind of fluids appears for instance in animal blood flow, lubrication theory, polymer suspensions and liquid crystals [19, 23, 27]. Now, among the sciences we have the following: physics, partial differential equations, functional analysis, control theory, numerical analysis, biology, hydrodynamics, etc [1, 3, 4, 20]. It is known that there exists several theories to describe the behavior of this kind of fluids but there is not still a universal theory to describe all of them. In particular, one of the most important approaches has been done by A. C. Eringen in [12] (see also [13]), where two relevant facts are introduced. First, it is introduced the concept of micropolar fluids to characterize the fluids consisting of rigid, randomly oriented (or spherical) particles suspended in a viscous medium, where the deformation of fluid particles is ignored. Second, it is deduced the mathematical model for micropolar fluids which consists of a simple and consistent generalization of the classical Navier-Stokes model. Nowadays, the micropolar fluids are also called asymmetric fluids and the mathematical properties are largely studied by several authors, see for instance [3, 5, 6, 7, 8, 10, 24, 25, 28, 32]. However, since the nonhomogeneous asymmetric model is a system related with Navier-Stokes equations, where several open questions exists. In particular, in this paper we address an answer to the question of well posedness for a source inverse problem. Moreover, for details of some other still unsolved problems related with control and a geometric inverse problems consult the recent survey given by E. Fernández-Cara and collaborators in [15].

In this paper we study the inverse problem of determining the density functions  $\mathbf{F}$  and  $\mathbf{G}$ , modelling the vector external sources for the linear and the angular momentum of particles, in a system for the motion in a finite time  $t \in [0, T]$  of nonhomogeneous viscous incompressible asymmetric fluid on a bounded and regular domain  $\Omega \subset \mathbb{R}^3$ , with boundary  $\partial\Omega$ :

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - (\mu + \mu_r) \Delta \mathbf{u} + \nabla p = 2\mu_r \operatorname{curl} \mathbf{w} + \rho \mathbf{F}, \quad \text{in } Q_T := \Omega \times [0, T], \quad (1.1)$$

$$\operatorname{div}(\mathbf{u}) = 0, \quad \text{in } Q_T, \quad (1.2)$$

$$(\rho \mathbf{w})_t + \operatorname{div}(\rho \mathbf{w} \otimes \mathbf{w}) - (c_a + c_d) \Delta \mathbf{w} - (c_0 + c_d - c_a) \nabla \operatorname{div} \mathbf{w} + 4\mu_r \mathbf{w}$$

*Date:* December 17, 2014.

*Key words and phrases.* inverse source problem, nonhomogeneous asymmetric fluids, micropolar fluids, integral overdetermination, navier-stokes system.

<sup>†</sup> GMA, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Chillán, Chile, E-mail: [acoronel@ubiobio.cl](mailto:acoronel@ubiobio.cl), [marko.medar@gmail.com](mailto:marko.medar@gmail.com).

$$= 2\mu_r \operatorname{curl} \mathbf{u} + \rho \mathbf{G}, \quad \text{in } Q_T, \quad c_0 + c_d > c_a, \quad (1.3)$$

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0, \quad \text{in } Q_T, \quad (1.4)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{w}(\mathbf{x}, 0) = \mathbf{w}_0(\mathbf{x}), \quad \rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \quad \text{on } \Omega, \quad (1.5)$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{w}(\mathbf{x}, t) = 0, \quad \text{on } \Sigma_T := \partial\Omega \times [0, T], \quad (1.6)$$

$$\int_{\Omega} \rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t) \cdot \boldsymbol{\psi}^{\mathbf{u}}(\mathbf{x}) d\mathbf{x} = \phi^{\mathbf{u}}(t), \quad \int_{\Omega} \rho(\mathbf{x}, t) \mathbf{w}(\mathbf{x}, t) \cdot \boldsymbol{\psi}^{\mathbf{w}}(\mathbf{x}) d\mathbf{x} = \phi^{\mathbf{w}}(t), \quad t \in [0, T]. \quad (1.7)$$

Here  $\mathbf{u}, \mathbf{w}, \rho$  and  $p$  denote the velocity field, the angular velocity of rotation of the fluid particles, the mass density and the pressure distribution, respectively. The constant  $\mu > 0$  is the usual Newtonian viscosity and the positive constants  $\mu_r, c_0$  and  $c_d$  are the additional viscosities related to the lack of symmetry of the stress tensor. The functions  $\boldsymbol{\psi}^{\mathbf{u}}, \boldsymbol{\psi}^{\mathbf{w}}, \phi^{\mathbf{u}}$  and  $\phi^{\mathbf{w}}$  in the integral overdetermination condition (1.7) are given and satisfy some restrictions which will be specified later. The differential notation is the standard ones, i.e. the symbols  $\mathbf{u}_t, \mathbf{w}_t$  and  $\rho_t$  denote the time derivatives and  $\nabla, \Delta, \operatorname{div}$  and  $\operatorname{curl}$  denote the gradient, Laplacian, divergence and rotational operators, respectively. Now, by applying the Helmholtz decomposition to  $\mathbf{F}$  and assuming a coherent form of  $\mathbf{G}$  we deduce that the vector fields  $\mathbf{F}$  and  $\mathbf{G}$  are representable by the following relations

$$\mathbf{F}(\mathbf{x}, t) = f(t)(\nabla h(\mathbf{x}, t) - \mathbf{m}(\mathbf{x}, t)), \quad \mathbf{G}(\mathbf{x}, t) = g(t)\mathbf{q}(\mathbf{x}, t), \quad \text{in } \Omega_T, \quad (1.8)$$

where  $\mathbf{m}$  and  $\mathbf{q}$  are given functions and  $f, g$  and  $h$  are unknown functions such that

$$\operatorname{div}(\rho \nabla h) = \operatorname{div}(\rho \mathbf{m}), \quad \text{in } \Omega, \quad (1.9)$$

$$\frac{\partial h}{\partial \mathbf{n}} = \mathbf{m} \cdot \mathbf{n}, \quad \text{on } \Sigma_T, \quad (1.10)$$

$$\int_{\Omega} h(\mathbf{x}, t) d\mathbf{x} = 0, \quad t \in [0, T], \quad (1.11)$$

where  $\mathbf{n}$  is the outward unit normal vector to  $\partial\Omega$ . We note that, the determination of  $f$  and  $g$ , called the coefficients of  $\mathbf{F}$  and  $\mathbf{G}$ , solves the inverse problem, since  $\mathbf{m}$  and  $\mathbf{q}$  are known. Indeed, if  $f$  is determined, we can find  $h$  by solving (1.9)-(1.11), since  $\mathbf{m}$  is given. Now, if additionally  $g$  is determined, it is clear that  $\mathbf{F}$  and  $\mathbf{G}$  can be recovered by (1.8). Hence, from (1.8), the inverse problem of recovery the vector fields  $\mathbf{F}$  and  $\mathbf{G}$  admits an equivalent interpretation like an inverse coefficients determination problem. Moreover, this behavior implies that the inverse problem can be equivalently reformulated as an operator equation of second kind (see subsection 5.1).

Similar inverse problems have been extensively studied by Prilepko, Orlovsky, Vasin and collaborators, see the monograph [26] and references therein. In a broad sense, their methodology to analyze the different source inverse problems consists of three big steps:

E<sub>1</sub>. To introduce the functional framework where the direct problem is well-posed.

E<sub>2</sub>. To use the definition of the solution of the direct problem, in order to derive an operator equation of the second kind which solvability is equivalent to the solution of the inverse problem.

E<sub>3</sub>. To prove the unique solution of the operator equation by applying fixed point arguments.

For instance, in order to fix ideas, let us consider the following inverse source problem for the heat equation: given  $g, \omega$  and  $\phi$  find a pair of functions  $\{u, f\}$  such that

$$\begin{cases} u_t(x, t) - \Delta u(x, t) &= f(t)g(x, t), & (x, t) \in Q_T, \\ u(x, 0) &= 0, & x \in \Omega, \\ u(x, t) &= 0, & (x, t) \in \partial\Sigma_T, \\ \int_{\Omega} u(x, t)w(x)dx &= \phi(x), & t \in [0, T]. \end{cases} \quad (1.12)$$

In this case for the step E<sub>1</sub>, is well known that one of the most general functional frameworks where the well-posedness of the forward problem holds is given by the Lebesgue and Sobolev spaces [11], see subsection 2.1 or consult [21, 22, 26] for the standard notation of this spaces. Actually, considering that  $g \in C([0, T], L^2(\Omega))$  and  $f \in L^2(\Omega)$ , we can prove that there exists  $u \in W_{2,0}^{2,1}(Q_T)$  a unique generalized solution of the forward problem (1.12) (i.e. except the overdetermination condition). In the second step E<sub>2</sub>, we prove that the solvability of the inverse problem (1.12) is

equivalent to the solution of the second kind operator equation  $f = Af + \psi$  on  $L^2(0, T)$ , where the operator  $A$  and the function  $\psi$  are defined as follows

$$(Af)(t) = \frac{1}{g_1(t)} \int_{\Omega} u(x, t) \Delta w(x) dx, \text{ and } \psi = \phi' g_1 \text{ with } g_1(t) = \int_{\Omega} g(x, t) w(x) dx. \quad (1.13)$$

Now, concerning to the step  $E_3$  using the fixed point arguments prove that  $f = Af + \psi$  has a unique solution. In point of fact, we can prove that the operator equation can be rewritten as follows  $f = \hat{A}f$  with  $\hat{A}f = Af + \phi'/g_1$  and  $\hat{A}$  has a fixed point in  $L^2(0, T)$ . In general, the fact  $E_3$  is a straightforward consequence of a priori estimates for the solution of the direct problem. Clearly, the central difficulty in the analysis of source problems, by applying this approach, is the proof of that the operator satisfies the hypothesis of the fixed point theorem (see Theorem 5.3). Now, we note that, it was proved in [26] that this general methodology can be applied to analyze the inverse source problems for elliptic, hyperbolic, parabolic and even for nonstationary linearized Navier-Stokes system with constant density (in this case also consult [31]). Recently, Fan and Nakamura in [14], following the same general approach and the results of J. Simon [29], have been proved the local solvability of the source problem for Navier-Stokes system with variable density function. In the present paper, we generalize these results for the inverse problem (1.1)-(1.11).

Other general approaches applied to close inverse problems are given in the following fundamental books on inverse problems [2, 17, 18]. In our knowledge many of these techniques have not been applied to analyze inverse source problems for Navier-Stokes and related systems. An exception, on this sense is the recent results obtained by Choulli and collaborators in [9] for the case of the non-stationary linearized NavierStokes equations where as observation data it is assumed that the velocity is given on an arbitrarily fixed sub-domain and over some time interval. We remark that on [9], in order to solve the inverse problem, the authors have used an approach based on Carleman estimates which was largely applied to geometric inverse problems [15].

The paper is organized as follows. In section 2, we recall some preliminary concepts and cover the step  $E_1$  of the methodology. In section 3, we develop some a priori estimates which will be useful in the step  $E_3$ . In section 4, we prove the well posedness of the direct problem, i.e. we develop the step  $E_2$ . Finally, in section 5 we set the details of the step  $E_3$ , obtaining the local solvability of the inverse problem (1.1)-(1.11).

## 2. PRELIMINARIES

In this section we consider the notation of the functional framework, the rigorous definition of direct and inverse problems and the general assumptions.

**2.1. Some functional spaces.** Let us starting by recalling the standard notation of some functional spaces and operators which are familiar in the mathematical theory of fluids modelled by Navier-Stokes system, see [3, 21, 22, 30]. The Banach space of measurable functions that are  $p$ -integrable in the sense of Lebesgue or are essentially bounded on  $\Omega$  are denoted by  $L^p(\Omega)$  for  $p \in [1, \infty[$  and by  $L^\infty(\Omega)$ , respectively. We recall that, the norms in  $L^p(\Omega)$  for  $p \in [1, \infty[$  and  $p = \infty$  are defined as follows

$$\|u\|_{L^p(\Omega)} := \left( \int_{\Omega} |u(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} \text{ and } \|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{\mathbf{x} \in \Omega} |u(\mathbf{x})|,$$

respectively. The notation  $W^{m,q}(\Omega)$ , where  $m \in \mathbb{N}$  and  $q \geq 1$  is used for the Sobolev space consisting of all functions in  $L^q(\Omega)$  having all distributional derivatives of the first  $m$  orders belongs to  $L^q(\Omega)$ , i.e.

$$W^{m,q}(\Omega) := \left\{ u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega) \text{ for } |\alpha| = 1, \dots, m \right\}.$$

The norm of  $W^{m,q}(\Omega)$  is naturally defined as follows

$$\|u\|_{W^{m,q}(\Omega)} := \left( \sum_{k=0}^m \sum_{|\alpha|=k} \|D^\alpha u\|_{L^q(\Omega)}^q \right)^{1/q} \text{ and } \|u\|_{W^{m,\infty}(\Omega)} := \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

The vector-valued spaces  $[L^2(\Omega)]^3$  and  $[W^{m,p}(\Omega)]^3$  are defined as usual and by simplicity are denoted by bold characters, i.e.  $\mathbf{L}^2(\Omega)$  and  $\mathbf{W}^{m,p}(\Omega)$ , respectively. Also, we use the following rather common notation in mathematical theory of fluid mechanics:

$$\begin{aligned} H^m(\Omega) &= W^{m,2}(\Omega), \quad H_0^m(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}, \\ \mathcal{V}(\Omega) &= \left\{ \mathbf{v} \in (C_0^\infty(\Omega))^3 : \operatorname{div}(\mathbf{v}) = 0 \text{ in } \Omega \right\}, \\ \mathbf{H} &= \overline{\mathcal{V}(\Omega)}^{\|\cdot\|_{\mathbf{L}^2(\Omega)}} \quad \text{and} \quad \mathbf{V} = \overline{\mathcal{V}(\Omega)}^{\|\cdot\|_{\mathbf{H}_0^1(\Omega)}}, \end{aligned}$$

where  $\overline{A}^{\|\cdot\|_B}$  denotes the closure of  $A$  in  $B$ . Furthermore, for a given Banach space  $X$ , we denote by  $L^r(0, T; X)$ ,  $r \geq 1$ , the Banach space of the  $X$ -valued functions having bounded the norm  $\|\cdot\|_{L^r(0, T; B)}$  defined as follows

$$\|u\|_{L^r(0, T; B)} := \left( \int_0^T \|u(\cdot, t)\|_B^r dt \right)^{1/r} \quad \text{and} \quad \|u\|_{L^\infty(0, T; B)} := \operatorname{ess\,sup}_{t \in [0, T]} \|u(\cdot, t)\|_B.$$

Concerning to the linear operators, we define the operators:  $A, L_0$  and  $L$ . We denote by  $A$  the stokes operator defined from  $D(A) := \mathbf{V} \cap \mathbf{H}^2(\Omega) \subset \mathbf{H}$  to  $\mathbf{H}$  by  $A\mathbf{v} = P(-\Delta\mathbf{v})$ , where  $P$  is the orthogonal projection of  $\mathbf{L}^2(\Omega)$  onto  $\mathbf{H}$  induced by the Helmholtz decomposition of  $\mathbf{L}^2(\Omega)$ . It is well known that  $A$  is an unbounded linear and positive self-adjoint operator, and is characterized by the following identity

$$(A\mathbf{w}, \mathbf{v}) = (\nabla\mathbf{w}, \nabla\mathbf{v}), \quad \forall \mathbf{w} \in D(A), \quad \mathbf{v} \in V, \quad (2.1)$$

where  $(\cdot, \cdot)$  is the usual scalar product in  $\mathbf{L}^2(\Omega)$ . In second place, we consider the strongly uniformly elliptic operators  $L_0$  and  $L$  defined on  $D(L_0) = D(L) = \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$  as follows

$$L_0 z = -(c_a + c_d)\Delta z - (c_0 + c_d - c_a)\nabla \operatorname{div} z \quad \text{and} \quad Lz = L_0 z + 4\mu_r z. \quad (2.2)$$

Note that  $L$  is a positive operator under the assumption  $c_0 + c_d > c_a$ , see (1.3).

**2.2. Direct and inverse problem solution definitions.** Using the described notation on the section 2.1 and the ideas given on [6], we can give a rigorous formulation of direct and inverse problem related to equations (1.1)-(1.11).

**Definition 2.1.** *Consider that the functions  $f, g, m$  and  $q$  are given. Then, a collection of functions  $\{\mathbf{u}, \mathbf{w}, \rho, p, h\}$  is a solution of the direct problem (1.1)-(1.5) and (1.8)-(1.11) if there exists  $T_* \in ]0, T]$  such that the functions satisfy the following four conditions*

(a) *Regularity conditions:*

$$\mathbf{u} \in C^0([0, T_*]; D(A)) \cap C^1([0, T_*]; \mathbf{H}), \quad (2.3)$$

$$\mathbf{w} \in C^0([0, T_*]; D(L)) \cap C^1([0, T_*]; \mathbf{L}^2(\Omega)) \quad \text{and} \quad (2.4)$$

$$\rho \in C^1(\overline{\Omega} \times [0, T_*]). \quad (2.5)$$

(b) *Integral identities:*

$$\begin{aligned} & \left( (\rho\mathbf{u})_t, \mathbf{v} \right) - \int_{\Omega} \rho\mathbf{u} \otimes \mathbf{u} : \nabla\mathbf{v} \, d\mathbf{x} + (\mu + \mu_r)(A\mathbf{u}, \mathbf{v}) \\ &= 2\mu_r \left( \operatorname{curl} \mathbf{w}, \mathbf{v} \right) + (\rho\mathbf{F}, \mathbf{v}), \quad \text{for } t \in ]0, T_*[ \text{ and } \forall \mathbf{v} \in V, \end{aligned} \quad (2.6)$$

$$\begin{aligned} & \left( (\rho\mathbf{w})_t, \boldsymbol{\varphi} \right) - \int_{\Omega} \rho\mathbf{w} \otimes \mathbf{w} : \nabla\boldsymbol{\varphi} \, d\mathbf{x} + (L\mathbf{w}, \boldsymbol{\varphi}) \\ &= 2\mu_r \left( \operatorname{curl} \mathbf{u}, \boldsymbol{\varphi} \right) + (\rho\mathbf{G}, \boldsymbol{\varphi}), \quad \text{for } t \in ]0, T_*[ \text{ and } \forall \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega). \end{aligned} \quad (2.7)$$

(c) *Mass conservation:*  $\rho$  satisfies (1.4) for  $(x, t) \in \overline{\Omega} \times [0, T_*]$ .

(d) *Initial condition:*  $\mathbf{u}, \mathbf{w}, \rho$  satisfies (1.5) for  $\mathbf{x} \in \Omega$ .

**Definition 2.2.** Consider that the functions  $\phi^{\mathbf{u}}, \phi^{\mathbf{w}}, \psi^{\mathbf{u}}, \psi^{\mathbf{w}}$   $m$  and  $q$  are given. Then, a collection of functions  $\{\mathbf{u}, \mathbf{w}, \rho, p, h, f, g\}$  is called a solution of the inverse problem (1.1)-(1.11) if the following three conditions hold:

- (i) The functions  $f$  and  $g$  are belongs to  $H^1([0, T])$ ,
- (ii) The collection  $\{\mathbf{u}, \mathbf{w}, \rho, p, h\}$  is a solution of the direct problem (1.1)-(1.5) and (1.8)-(1.11) and
- (iii) The overdetermination condition (1.7) is satisfied.

**2.3. General hypothesis and well-posedness of the direct problem.** Hereafter, we make the following regularity assumptions:

- (H<sub>1</sub>) The initial density  $\rho_0$  belongs to  $C^1(\overline{\Omega})$  and  $\rho_0(\mathbf{x}) \in [\alpha, \beta] \subset ]0, \infty[$  a.e. in  $\Omega$ ,
- (H<sub>2</sub>) The initial velocity  $\mathbf{u}_0$  belongs to  $D(A)$  and
- (H<sub>3</sub>) The initial angular velocity  $\mathbf{w}_0$  belongs to  $D(L)$ .
- (H<sub>4</sub>) The functions  $h$  and  $r$  belong to  $C^1([0, T])$ .
- (H<sub>5</sub>) The functions  $\psi^{\mathbf{u}}$  and  $\psi^{\mathbf{w}}$  belong to  $\mathbf{H}_0^1(\Omega) \cap \mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{H}^2(\Omega)$  and such that  $\text{div}(\psi^{\mathbf{u}}) = 0$  in  $\Omega$ .
- (H<sub>6</sub>) The functions  $\phi^{\mathbf{u}}$  and  $\phi^{\mathbf{w}}$  belong to  $(H^2(\Omega))^3$ .
- (H<sub>7</sub>) There exists  $h^\epsilon, r^\epsilon \in \mathbb{R}^+$  such that

$$\left| \int_{\Omega} \rho_0(\mathbf{x}) (\nabla h(\mathbf{x}, 0) - \mathbf{m}(\mathbf{x}, 0)) d\mathbf{x} \right| \geq 2h^\epsilon \quad \text{and} \quad \left| \int_{\Omega} \rho_0(\mathbf{x}) \mathbf{q}(\mathbf{x}, 0) d\mathbf{x} \right| \geq 2r^\epsilon.$$

### 3. A PRIORI ESTIMATES

The main result of this section is the following theorem:

**Theorem 3.1.** Let  $\eta \in \mathbb{R}^+$  such that  $\|(f, g)\|_{[H^1(0,T)]^2} \leq \eta$ . Then, there exists  $\kappa_j \in \mathbb{R}^+$  for  $j = 1, \dots, 11$  and two small enough times  $T_1, T_2 \in [0, T_*]$ , independents of  $f$  and  $g$ , such that the following estimates hold

$$\|\mathbf{u}\|_{L^\infty([0, T_1]; \mathbf{H}_0^1(\Omega))} + \|\mathbf{w}\|_{L^\infty([0, T_1]; \mathbf{H}_0^1(\Omega))} + \|\mathbf{u}_t\|_{L^\infty([0, T_1]; \mathbf{L}^2(\Omega))} + \|\mathbf{w}_t\|_{L^\infty([0, T_1]; \mathbf{L}^2(\Omega))} \leq \kappa_1, \quad (3.1)$$

$$\begin{aligned} \|\mathbf{u}_t\|_{L^\infty([0, T_2]; \mathbf{L}^2(\Omega))} + \|\mathbf{w}_t\|_{L^\infty([0, T_2]; \mathbf{L}^2(\Omega))} + \|\mathbf{u}_t\|_{L^2([0, T_2]; \mathbf{H}^1(\Omega))} \\ + \|\mathbf{w}_t\|_{L^2([0, T_2]; \mathbf{H}^1(\Omega))} + \|\nabla \rho\|_{L^\infty([0, T_*]; \mathbf{L}^q(\Omega))} \leq \kappa_2, \end{aligned} \quad (3.2)$$

$$\|h\|_{L^\infty([0, T_2]; H^2(\Omega))} \leq \kappa_3, \quad (3.3)$$

$$\|\nabla h_t\|_{L^\infty([0, T_2]; \mathbf{L}^2(\Omega))} \leq \kappa_4 + \kappa_5 \eta, \quad (3.4)$$

$$\|\mathbf{u}\|_{L^\infty([0, T_1]; \mathbf{H}^2(\Omega))} + \|p\|_{L^\infty([0, T_1]; H^1(\Omega))} \leq \kappa_6 + \kappa_7 \eta, \quad (3.5)$$

$$\|\rho_t\|_{L^\infty([0, T_*]; \mathbf{L}^2(\Omega))} \leq \kappa_8, \quad \|\rho_t\|_{L^\infty([0, T_*]; \mathbf{L}^q(\Omega))} \leq \kappa_9 + \kappa_{10} \eta \quad \text{and} \quad (3.6)$$

$$\|\mathbf{u}\|_{L^2([0, T_2]; \mathbf{W}^{2,s}(\Omega))} + \|p\|_{L^2([0, T_2]; W^{1,s}(\Omega))} \leq \kappa_{11}. \quad (3.7)$$

We postpone the proof of Theorem 3.1 to the subsection 3.2, since we require the introduction of several appropriate space estimates which leads to the inequalities (3.1)-(3.7).

**3.1. Some space a priori estimates.** We recall that, by Pioncaré and the Gagliardo-Nirenberg inequalities we have that there exists  $C_{poi} > 0$  and  $C_{gn} > 0$  depending only on  $p, q$  and  $\Omega$  such that

$$\begin{aligned} \|u\|_{L^q(\Omega)} &\leq C_{poi} \|\nabla u\|_{L^p(\Omega)}, \quad p \in [1, 3[, \quad q \in [1, 3p(3-p)^{-1}] \quad \text{and} \quad u \in W_0^{1,p}(\Omega), \\ \|\nabla u\|_{L^{2q/(q-2)}(\Omega)} &\leq C_{gn} \|\nabla u\|_{L^2(\Omega)}^{1-3/q} \|u\|_{H^2(\Omega)}^{3/q}, \quad u \in H^2(\Omega). \end{aligned} \quad (3.8)$$

Also we denote by  $C_{iny}^{2,\infty}$  the constant such that  $\|u\|_{L^\infty(\Omega)} \leq C_{iny}^{2,\infty} \|u\|_{H^2(\Omega)}$  for all  $u \in H^2(\Omega)$ , i.e. for the continuous embedding of  $H^2(\Omega)$  in  $L^\infty(\Omega)$ .

**Lemma 3.1.** The following estimate holds:  $0 < \alpha \leq \rho(\mathbf{x}, t) \leq \beta$  for all  $(\mathbf{x}, t) \in \Omega \times [0, T_*]$ .

*Proof.* We deduce the estimate by equations (1.2) and (1.4), the hypothesis (H<sub>1</sub>) and the maximum principle.  $\square$

**Lemma 3.2.** Assume that  $\{\epsilon, \epsilon', \epsilon'', \dot{\epsilon}\} \subset \mathbb{R}^+$  with  $\epsilon \in ]0, 2\alpha\beta^{-1}[$  and  $\epsilon' \in ]0, \alpha(C^{reg}C_{gn})^{-1}[$ . Then, there exists  $\Pi_i$ ,  $i = 1, \dots, 4$ , defined as follows

$$\Pi_1 = \sqrt{\beta\epsilon^{-1}(2\alpha - \beta\epsilon)^{-1}}, \quad (3.9)$$

$$\Pi_2 = \frac{C^{reg}}{\alpha - \epsilon' C^{reg} C_{gn}} \left( \alpha \|\nabla \mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} + |\Omega|^{(q-2)/q} \epsilon' \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty} \right), \quad (3.10)$$

$$\Pi_3 = \frac{C^{reg}}{\alpha - \epsilon' C^{reg} C_{gn}} \left( |\Omega|^{(q-2)/q} \epsilon' (\epsilon'')^{3/(3-q)} + C_{gn} (\epsilon')^{3/(3-q)} \Pi_1 \|\nabla \mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \right), \quad (3.11)$$

$$\Pi_4 = \frac{C_{gn} C_{iny}^{2,\infty} \dot{\epsilon}}{\alpha} \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \quad \text{and} \quad (3.12)$$

$$\Pi_5 = \frac{C_{iny}^{2,\infty}}{\alpha} \left( C_{gn} (\dot{\epsilon})^{3/(3-q)} \Pi_1 \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} + C_{poi} \|\mathbf{m}(\cdot, t)\|_{\mathbf{H}^2} \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \right), \quad (3.13)$$

such that the following estimates holds

$$\|\nabla h(\cdot, t)\|_{\mathbf{L}^2} \leq \Pi_1 \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2}, \quad (3.14)$$

$$\|h(\cdot, t)\|_{H^2} \leq \Pi_2 + \Pi_3 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^{q/(q-3)}, \quad (3.15)$$

$$\|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} \leq \frac{\beta}{\alpha} \|\mathbf{m}_t(\cdot, t)\|_{\mathbf{L}^2} + \Pi_4 \left( \Pi_2 + \Pi_3 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^{q/(q-3)} \right) + \Pi_5 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}, \quad (3.16)$$

for all  $t \in [0, T_*]$ . Hereafter, the notation  $\|\cdot\|_{L^p}$  and  $\|\cdot\|_{H^p}$  are used to abbreviate the norm  $\|\cdot\|_{L^p(\Omega)}$  and  $\|\cdot\|_{H^p(\Omega)}$ , respectively.

*Proof.* Form (1.9), by applying Lemma 3.1, integration by parts, the boundary condition (1.10), and Hölder and Young inequalities, we have that

$$\begin{aligned} \alpha \int_{\Omega} |\nabla h|^2(\mathbf{x}, t) d\mathbf{x} &\leq \int_{\Omega} \left( \rho |\nabla h|^2 \right)(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} \left( \rho \nabla h \cdot \mathbf{m} \right)(\mathbf{x}, t) d\mathbf{x} \\ &\leq \beta \|\nabla h(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \leq \frac{\epsilon\beta}{2} \|\nabla h(\cdot, t)\|_{\mathbf{L}^2}^2 + \frac{\beta}{2\epsilon} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2}^2, \end{aligned}$$

for each  $t \in [0, T_*]$ . Hence, we see that (3.14) holds for  $\Pi_1$  defined by (3.9).

Now, we can proceed to prove (3.15). We start recalling the identities  $\operatorname{div}(\rho \nabla h) = \rho \Delta h + \nabla \rho \cdot \nabla h$  and  $\operatorname{div}(\rho \mathbf{m}) = \rho \operatorname{div}(\mathbf{m}) + \nabla \rho \cdot \mathbf{m}$ , which imply that the equation (1.9) can be rewritten as follows

$$\Delta h = \operatorname{div}(\mathbf{m}) + \frac{1}{\rho} \nabla \rho \cdot \mathbf{m} - \frac{1}{\rho} \nabla \rho \cdot \nabla h. \quad (3.17)$$

Clearly, by the estimate (3.14) we deduce that the right hand side of (3.17) belongs to  $L^2(\Omega)$ . Then, by the regularity of solutions for (3.17), the inequality (3.8), Lemma 3.1 and the estimate (3.14), we can follow that there exists  $C_{reg} > 0$  independent of  $h$  such that the following bound

$$\begin{aligned} \|h(\cdot, t)\|_{H^2} &\leq C_{reg} \left\| \left( \operatorname{div}(\mathbf{m}) + \frac{1}{\rho} \nabla \rho \cdot \mathbf{m} - \frac{1}{\rho} \nabla \rho \cdot \nabla h \right)(\cdot, t) \right\|_{\mathbf{L}^2} \\ &\leq C_{reg} \left\{ \|\nabla \mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} + \frac{1}{\alpha} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^2} + \frac{1}{\alpha} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\nabla h(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \right\} \\ &\leq C_{reg} \left\{ \|\nabla \mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} + \frac{|\Omega|^{\frac{q-2}{2q}}}{\alpha} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \right. \\ &\quad \left. + \frac{C_{gn}}{\alpha} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\nabla h(\cdot, t)\|_{\mathbf{L}^2}^{1-3/q} \|h(\cdot, t)\|_{H^2}^{3/q} \right\} \\ &\leq C_{reg} \left\{ \|\nabla \mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} + \frac{|\Omega|^{\frac{q-2}{2q}}}{\alpha} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \right. \\ &\quad \left. + \frac{C_{gn}(\Pi_1)^{1-3/q}}{\alpha} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2}^{1-3/q} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|h(\cdot, t)\|_{H^2}^{3/q} \right\}, \end{aligned}$$

holds for each  $t \in [0, T_*]$ . Thus, by the application of two times of the Young's inequality we complete the proof of (3.15) with  $\Pi_2$  of the form given in (3.10).

The proof of (3.16) is given as follows. Taking  $\partial_t$  to the first equation of (1.9), testing the result by  $h_t$ , using the estimate of Lemma 3.1, the Hölder inequality, the equation (1.4) and inequality (3.8), we have that

$$\begin{aligned}
\alpha \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2}^2 &\leq \int_{\Omega} \rho |\nabla h_t(\cdot, t)|^2 d\mathbf{x} \\
&\leq \left| \int_{\Omega} (\rho \mathbf{m}_t \cdot \nabla h_t)(\cdot, t) d\mathbf{x} \right| + \left| \int_{\Omega} (\rho_t \nabla h \cdot \nabla h_t)(\cdot, t) d\mathbf{x} \right| + \left| \int_{\Omega} (\rho_t \mathbf{m} \cdot \nabla h_t)(\cdot, t) d\mathbf{x} \right| \\
&\leq \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^\infty} \|\mathbf{m}_t(\cdot, t)\|_{\mathbf{L}^2} \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} + \left| \int_{\Omega} ((\mathbf{u} \cdot \nabla \rho) \nabla h \cdot \nabla h_t)(\cdot, t) d\mathbf{x} \right| \\
&\quad + \left| \int_{\Omega} ((\mathbf{u} \cdot \nabla \rho) \mathbf{m} \cdot \nabla h_t)(\cdot, t) d\mathbf{x} \right| \\
&\leq \beta \|\mathbf{m}_t(\cdot, t)\|_{\mathbf{L}^2} \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} + \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\nabla h(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} \\
&\quad + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} \\
&\leq \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} \left\{ \beta \|\mathbf{m}_t(\cdot, t)\|_{\mathbf{L}^2} + C_{gn} \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\nabla h(\cdot, t)\|_{\mathbf{L}^2}^{1-3/q} \|h(\cdot, t)\|_{\mathbf{H}^2}^{3/q} \right. \\
&\quad \left. + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \right\}.
\end{aligned}$$

Then, by (3.8), the continuous inclusion of  $H^2$  in  $L^\infty$ , and the Young inequality, we obtain

$$\begin{aligned}
\|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} &\leq \frac{\beta}{\alpha} \|\mathbf{m}_t(\cdot, t)\|_{\mathbf{L}^2} + \frac{C_{gn} C_{iny}^{2,\infty} \dot{\epsilon}}{\alpha} \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|h(\cdot, t)\|_{\mathbf{H}^2} + \frac{C_{iny}^{2,\infty}}{\alpha} \times \\
&\quad \left( C_{gn} (\dot{\epsilon})^{3/(3-q)} \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2} \|\nabla h(\cdot, t)\|_{\mathbf{L}^q} + C_{poi} \dot{\epsilon} \|\mathbf{m}(\cdot, t)\|_{\mathbf{H}^2} \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \right) \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q},
\end{aligned}$$

which implies (3.16) by straightforward application of (3.14) and (3.15).  $\square$

**Lemma 3.3.** *There exists  $\Upsilon_i \in \mathbb{R}^+$  for  $i \in \{1, \dots, 5\}$ , depending only on  $\Omega, c_a, c_0, c_d, \alpha, \beta$  and  $\mu_r$  (independent of  $f$  and  $g$ ), such that the following estimate holds:*

$$\begin{aligned}
&\Upsilon_1 \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2} + \|p(\cdot, t)\|_{\mathbf{H}^2} + \Upsilon_1 \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2} \\
&\leq \Upsilon_2 \left[ \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^3 + \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^3 \right] + \Upsilon_3 \left[ \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} + \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \right] \\
&\quad + \Upsilon_4 \left[ |f(t)| \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2(\Omega)} + |g(t)| \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2(\Omega)} \right] \\
&\quad + \Upsilon_5 \left[ \|(\sqrt{\rho} \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2} + \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2} \right], \tag{3.18}
\end{aligned}$$

for all  $t \in [0, T_*]$ .

*Proof.* The inequality (3.18) is a consequence of the regularity of solutions for the Stokes system satisfied by  $u$  and  $p$  and the uniformly elliptic equation satisfied by  $w$ . Indeed, we first note that the equations (1.1), (1.2) and (1.8) imply that  $\mathbf{u}$  and  $p$  satisfy the Stokes problem given by the equation

$$-(\mu + \mu_r) \Delta \mathbf{u} + \nabla p = 2\mu_r \operatorname{curl} \mathbf{w} + \rho f (\nabla h - \mathbf{m}) - \rho \mathbf{u}_t - \rho (\mathbf{u} \cdot \nabla) \mathbf{u}, \quad \text{in } Q_T, \tag{3.19}$$

where the incompressibility condition is given by (1.2) and the initial and boundary conditions are given by (1.5) and (1.6), respectively. Hence, by applying the result given in [30] for the regularity of the solutions for stokes equation, the Minkowski and Hölder inequalities, we deduce that

$$\begin{aligned}
&\|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2} + \|p(\cdot, t)\|_{\mathbf{H}^2} \\
&\leq C_1^{reg} \left[ 2\mu_r \|\operatorname{curl} \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} + \|(\rho f (\nabla h - \mathbf{m}))(\cdot, t)\|_{\mathbf{L}^2} + \|(\rho \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2} \right. \\
&\quad \left. + \|\rho ((\mathbf{u} \cdot \nabla) \mathbf{u})(\cdot, t)\|_{\mathbf{L}^2} \right] \\
&\leq C_1^{reg} \left[ 2\mu_r \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} + |f(t)| \|\rho(\cdot, t)\|_{\mathbf{L}^2} \left( \|\nabla h(\cdot, t)\|_{\mathbf{L}^2} + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \right) \right. \\
&\quad \left. + \|(\rho \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2} + \|\rho(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^3} \right], \tag{3.20}
\end{aligned}$$



where  $C_1^{reg}$  is a positive constant depending on  $\mu, \mu_r$  and  $\Omega$ . In the second place, by (1.3), (1.8) and (2.2), we deduce that  $\mathbf{w}$  satisfies the following equation

$$L\mathbf{w} = 2\mu_r \operatorname{curl} \mathbf{u} + \rho g \mathbf{q} - \rho \mathbf{u}_t - \rho(\mathbf{w} \cdot \nabla) \mathbf{w}, \quad \text{in } Q_T. \quad (3.21)$$

Then by the regularity results for the solutions for uniformly elliptic equations (see for instance [11]), the Minkowski and Hölder inequalities, and (3.8), we have that

$$\begin{aligned} \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2} &\leq C_2^{reg} \left[ 2\mu_r \|\operatorname{curl} \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} + \|(\rho g \mathbf{q})(\cdot, t)\|_{\mathbf{L}^2} + \|(\rho \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2} \right. \\ &\quad \left. + \|\rho((\mathbf{w} \cdot \nabla) \mathbf{w})(\cdot, t)\|_{\mathbf{L}^2} + \|\mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} \right] \\ &\leq C_2^{reg} \left[ 2\mu_r \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} + |g(t)| \|\rho(\cdot, t)\|_{L^2} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} + \|(\rho \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2} \right. \\ &\quad \left. + \|\rho(\cdot, t)\|_{L^2} \|\mathbf{w}(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^3} + C_{poi} \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} \right], \end{aligned} \quad (3.22)$$

where  $C_2^{reg}$  is a positive constant depending only on  $\Omega$  and on the coefficients of  $L$ . Now, we note that the second terms on the right hand sides of (3.20) and (3.22) can be bound by application of Lemmas 3.1-3.2 and (3.8). Hence, if we sum the bounded results, we obtain the following inequality

$$\begin{aligned} &\|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2} + \|p(\cdot, t)\|_{H^2} + \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2} \\ &\leq C_M \left\{ (2\mu_r + C_{poi}) \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} + 2\mu_r \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \right. \\ &\quad \left. + \beta |\Omega|^{1/2} (\Pi_1 + 1) \left[ |f(t)| \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} + |g(t)| \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} \right] \right. \\ &\quad \left. + (\beta)^{1/2} \left[ \|(\sqrt{\rho} \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2} + \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2} \right] \right. \\ &\quad \left. + \beta |\Omega|^{1/2} C_{gn} C_{poi} \left[ \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^{3/2} \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2}^{1/2} + \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^{3/2} \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2}^{1/2} \right] \right\}, \end{aligned} \quad (3.23)$$

for all  $t \in [0, T_*]$  with  $C_M = \max\{C_1^{reg}, C_2^{reg}\}$ . Now, for  $\epsilon^* \in \mathbb{R}^+$  we define  $\Upsilon_i$  for  $i = 1, \dots, 5$  as follows

$$\begin{aligned} \Upsilon_1 &= (\epsilon^*)^{-2} \left( (\epsilon^*)^2 - \Upsilon_2 \right), \quad \Upsilon_2 = 2^{-1} \beta |\Omega|^{1/2} C_{gn} C_{poi} C_M \epsilon^*, \quad \Upsilon_3 = (2\mu_r + C_3) C_M, \\ \Upsilon_4 &= \beta |\Omega|^{1/2} (\Pi_1 + 1) C_M, \quad \Upsilon_5 = (\beta)^{1/2} C_M. \end{aligned}$$

Thus, selecting  $\epsilon^*$  such that  $(\epsilon^*)^2 > \Upsilon_2$  and applying the Young inequality to the last two terms of (3.23), we get (3.18).  $\square$

**Lemma 3.4.** *There exists  $T_1 \in [0, T_*]$  and  $\Theta : [0, T_1] \rightarrow \mathbb{R}^+$  independents of  $f$  and  $g$  such that the following estimate holds*

$$\|\mathbf{u}(\cdot, t)\|_{\mathbf{H}_0^1} + \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}_0^1} \leq \Theta(t), \quad (3.24)$$

for all  $t \in [0, T_1]$ .

*Proof.* Testing the equations (1.1) and (1.3) by  $\mathbf{u}_t$  and  $\mathbf{w}_t$ , respectively; summing the results; and applying the Minkowski and Hölder inequalities, we get

$$\begin{aligned} &\frac{(\mu + \mu_r)}{2} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x} + \frac{(c_0 + 2c_d)}{2} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{w}(\mathbf{x}, t)|^2 d\mathbf{x} + \int_{\Omega} (\rho |\mathbf{u}_t|^2)(\mathbf{x}, t) d\mathbf{x} \\ &\quad + \int_{\Omega} (\rho |\mathbf{w}_t|^2)(\mathbf{x}, t) d\mathbf{x} = - \int_{\Omega} (\rho(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + 2\mu_r \int_{\Omega} (\operatorname{curl} \mathbf{w} \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} \\ &\quad + \mu_r f(t) \int_{\Omega} (\rho(\nabla h - \mathbf{m}) \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} - \int_{\Omega} (\rho(\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \\ &\quad + 2\mu_r \int_{\Omega} (\operatorname{curl} \mathbf{u} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} + \mu_r g(t) \int_{\Omega} (\rho \mathbf{q} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \\ &\leq \left[ \|(\rho \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^3} + \|(\rho \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{w}(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^3} \right] \end{aligned}$$



$$\begin{aligned}
& + \left[ 2\mu_r \left\{ \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} + \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right\} \right] \\
& + \left[ |f(t)| \|(\rho \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2} \left( \|\nabla h(\cdot, t)\|_{\mathbf{L}^2} + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \right) + |g(t)| \|(\rho \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} \right] \\
& + \left[ 4\mu_r \|\mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right] := \sum_{i=1}^4 J_i, \tag{3.25}
\end{aligned}$$

where each  $J_i$  are defined by the corresponding brackets [ ]. Now, we will prove the estimate by getting some bounds for each  $J_i$  and then applying a Gronwall type inequality. Indeed, first, for  $J_1$ , by Lemma 3.1, inequality (3.8), Young inequality and Lemma 3.3, we deduce that

$$\begin{aligned}
J_1 & \leq \sqrt{\beta} C_{poi} C_{gn} \left[ \|(\sqrt{\rho} \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2} \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^{3/2} \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2}^{1/2} + \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2} \times \right. \\
& \quad \left. \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^{3/2} \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2}^{1/2} \right] \\
& \leq \frac{\sqrt{\beta} C_{poi} C_{gn}}{2} \left\{ \|(\sqrt{\rho} \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2} \left[ \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^3 + \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2}^3 \right] + \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2} \times \right. \\
& \quad \left. \left[ \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^3 + \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2}^3 \right] \right\} \\
& \leq \left( \frac{\Psi_1}{\epsilon^u} + \Psi_2 \right) \|(\sqrt{\rho} \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 + \left( \frac{\Psi_3}{\epsilon^w} + \Psi_2 \right) \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 + \Psi_4 \left[ \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^6 \right. \\
& \quad \left. + \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^6 \right] + \Psi_5 \left[ \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^2 \right] + \Psi_6 |f(t)|^2 + \Psi_7 |g(t)|^2 \tag{3.26}
\end{aligned}$$

where

$$\begin{aligned}
\Psi_1 & = \frac{\sqrt{\beta} C_{poi} C_{gn}}{4} \left( 1 + 2 \frac{\Upsilon_2}{\Upsilon_1} + 2 \frac{\Upsilon_3}{\Upsilon_1} + 2 \frac{\Upsilon_4}{\Upsilon_1} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \right), \quad \Psi_2 = \sqrt{\beta} C_{poi} C_{gn} \frac{\Upsilon_5}{\Upsilon_1}, \\
\Psi_3 & = \frac{\sqrt{\beta} C_{poi} C_{gn}}{4} \left( 1 + 2 \frac{\Upsilon_2}{\Upsilon_1} + 2 \frac{\Upsilon_3}{\Upsilon_1} + 2 \frac{\Upsilon_4}{\Upsilon_1} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} \right), \\
\Psi_4 & = \frac{\sqrt{\beta} C_{poi} C_{gn}}{4} \left( 1 + 2 \frac{\Upsilon_2}{\Upsilon_1} \right) \max\{\epsilon^u, \epsilon^w\}, \quad \Psi_5 = \frac{\sqrt{\beta} C_{poi} C_{gn}}{2} \frac{\Upsilon_3}{\Upsilon_1}, \\
\Psi_6 & = \frac{\sqrt{\beta} C_{poi} C_{gn}}{2} \frac{\Upsilon_4}{\Upsilon_1} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \quad \text{and} \quad \Psi_7 = \frac{\sqrt{\beta} C_{poi} C_{gn}}{2} \frac{\Upsilon_4}{\Upsilon_1} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2}.
\end{aligned}$$

Now, for  $J_i$ ,  $i = 2, 3, 4$ , by inequality (3.8), Lemmas 3.1-3.2 and Young inequality, we deduce that

$$\begin{aligned}
J_2 & \leq \frac{\mu_r}{\alpha} \left[ \frac{1}{\epsilon^u} \|(\sqrt{\rho} \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 + \frac{1}{\epsilon^w} \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 \right] \\
& \quad + \mu_r \max\{\epsilon^u, \epsilon^w\} \left[ \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^2 \right], \tag{3.27}
\end{aligned}$$

$$\begin{aligned}
J_3 & \leq \frac{\sqrt{\beta} (\|\nabla h(\cdot, t)\|_{\mathbf{L}^2} + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2})}{2 \epsilon^u} \|(\sqrt{\rho} \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 + \frac{\sqrt{\beta} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2}}{2 \epsilon^w} \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 \\
& \quad + \frac{\sqrt{\beta}}{2} \left( \|\nabla h(\cdot, t)\|_{\mathbf{L}^2} + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \right) \epsilon^u |f(t)|^2 + \frac{\sqrt{\beta}}{2} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} \epsilon^w |g(t)|^2 \\
& \leq \frac{\sqrt{\beta}}{2} (\Pi_1 + 1) \left\{ \frac{1}{\epsilon^u} \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \|(\sqrt{\rho} \mathbf{u}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 + \frac{1}{\epsilon^w} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 \right\} \\
& \quad + \frac{\sqrt{\beta}}{2} (\Pi_1 + 1) \left\{ \epsilon^u \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} |f(t)|^2 + \epsilon^w \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} |g(t)|^2 \right\}, \tag{3.28}
\end{aligned}$$

$$J_4 \leq \frac{2\mu_r C_{poi} \epsilon^w}{\alpha} \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^2 + \frac{2\mu_r C_{poi}}{\alpha \epsilon^w} \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2}^2. \tag{3.29}$$

Inserting (3.26)-(3.29) in (3.25) and selecting  $\epsilon^\ell > \max\{N^\ell(1 - \Psi_2)^{-1}, 0\}$  for  $\ell \in \{\mathbf{u}, \mathbf{w}\}$ , where

$$\begin{aligned}
N^{\mathbf{u}} & = \Psi_1 + \frac{\mu_r}{\alpha} + \frac{(\beta)^{1/2}}{2} (\Pi_1 + 1) \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \quad \text{and} \\
N^{\mathbf{w}} & = \Psi_2 + \frac{\mu_r}{\alpha} (2C_{poi} + 1) + \frac{(\beta)^{1/2}}{2} (\Pi_1 + 1) \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2},
\end{aligned}$$

we find that there exists  $\Xi_1, \Xi_2$  and  $\Xi_3$  defined as follows

$$\begin{aligned}\Xi_1 &= \Psi_4 \mathcal{C}^{-1}, & \Xi_2 &= \mathcal{C}^{-1} (\Psi_5 + \mu_r \epsilon^{\mathbf{w}} \alpha^{-1} (2C_{poi} + 1)), \\ \Xi_3 &= \mathcal{C}^{-1} \max \left\{ \Psi_6 + \epsilon^{\mathbf{u}} \|\mathbf{m}(\cdot, t)\|_{L^2}, \Psi_7 + \epsilon^{\mathbf{w}} \|\mathbf{q}(\cdot, t)\|_{L^2} \right\}\end{aligned}$$

with  $\mathcal{C} = \min\{2^{-1}(\mu + \mu_r), 2^{-1}(c_0 + 2c_d), 1 - \Psi_2 - (\epsilon^{\mathbf{u}})^{-1}N^{\mathbf{u}}, 1 - \Psi_2 - (\epsilon^{\mathbf{w}})^{-1}N^{\mathbf{w}}\}$  and such that the inequality

$$\begin{aligned}\frac{d}{dt} \left( \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^2 \right) + \|(\sqrt{\rho} \mathbf{v}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 + \|(\sqrt{\rho} \mathbf{w}_t)(\cdot, t)\|_{\mathbf{L}^2}^2 &\leq \Xi_1 \left[ \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^6 \right. \\ &\quad \left. + \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^6 \right] + \Xi_2 \left[ \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^2 \right] + \Xi_3 \left[ |f(t)|^2 + |g(t)|^2 \right]\end{aligned}\quad (3.30)$$

holds for  $t \in [0, T_*]$ . Now, making use of Lemma 3 given on [16], we conclude the existence of  $T_1$  depending on  $\|\nabla \mathbf{u}(\cdot, 0)\|_{\mathbf{L}^2}$  and  $\|\nabla \mathbf{w}(\cdot, 0)\|_{\mathbf{L}^2}$  such the estimate (3.24) holds with  $\Theta$  depending only on  $\Xi_1, \Xi_2, \Xi_3, \|\nabla \mathbf{u}(\cdot, 0)\|_{\mathbf{L}^2}$  and  $\|\nabla \mathbf{w}(\cdot, 0)\|_{\mathbf{L}^2}$ .  $\square$

**Lemma 3.5.** *Consider  $T_1$  as is given on Lemma 3.4. Then, there exists  $\Phi_i$ ,  $i = 1, \dots, 6$ , independent of  $f$  and  $g$  such that the following estimate holds*

$$\begin{aligned}&\frac{d}{dt} \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \frac{d}{dt} \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2 \\ &\leq \Phi_1 \left[ |f'(t)| \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} + |g'(t)| \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right] + \Phi_2 \left[ |f(t)| \left\{ \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} + 1 \right\} \times \right. \\ &\quad \left. \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} + |g(t)| \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right] + \Phi_3 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^2 \left[ |f(t)|^2 \left( \|h(\cdot, t)\|_{H^2}^2 + 1 \right) \right. \\ &\quad \left. + |g(t)|^2 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^2 \right] + \Phi_4 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^{4q/(3q-6)} \left[ \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{H}^2}^2 + \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{H}^2}^2 \right] \\ &\quad + \Phi_5 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^2 \left[ \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2}^{6/q} + \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2}^{6/q} \right] \\ &\quad + \Phi_6 \left[ \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2 \right],\end{aligned}\quad (3.31)$$

for all  $t \in [0, T_1]$ .

*Proof.* Differentiating (1.1) and (1.3) with respect to  $t$ ; testing the results by  $\mathbf{u}_t$  and  $\mathbf{w}_t$ , respectively; summing the resulting equations; and rearranging the terms we get

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\mathbf{u}_t(\mathbf{x}, t)|^2 d\mathbf{x} + (\mu + \mu_r) \int_{\Omega} |\nabla u_t(\mathbf{x}, t)|^2 d\mathbf{x} + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\mathbf{w}_t(\mathbf{x}, t)|^2 d\mathbf{x} \\ &\quad + (c_a + c_d) \int_{\Omega} |\nabla \mathbf{w}_t(\mathbf{x}, t)|^2 d\mathbf{x} = 2\mu_r \left[ \int_{\Omega} \text{curl} \cdot \mathbf{w}(x, t) \mathbf{u}_t(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} \text{curl} \mathbf{u}(x, t) \cdot \mathbf{w}_t(\mathbf{x}, t) d\mathbf{x} \right] \\ &\quad + \left[ \int_{\Omega} f'(t) (\rho (\nabla h - \mathbf{m}) \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} g'(t) (\rho \mathbf{q} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right] \\ &\quad + \left[ \int_{\Omega} f(t) (\rho (\nabla h_t - \mathbf{m}_t) \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} g(t) (\rho \mathbf{q}_t \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right] \\ &\quad + \left[ \int_{\Omega} f(t) (\rho_t (\nabla h - \mathbf{m}) \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} g(t) (\rho_t \mathbf{q} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right] \\ &\quad - \frac{1}{2} \left[ \int_{\Omega} (\rho_t |\mathbf{u}_t|^2)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (\rho_t |\mathbf{w}_t|^2)(\mathbf{x}, t) d\mathbf{x} \right] \\ &\quad + \left[ \int_{\Omega} (\rho_t (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (\rho_t (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right] \\ &\quad - \left[ \int_{\Omega} (\rho (\mathbf{u}_t \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (\rho (\mathbf{w}_t \cdot \nabla) \mathbf{w} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right] = \sum_{i=0}^6 I_i,\end{aligned}\quad (3.32)$$

where  $I_i$  for  $i = 0, \dots, 6$  are defined by the brackets. Hence, the proof of (3.31) is reduced to get some bounds for each  $I_i$  based on Minkowski and Hölder inequalities and the previous Lemmas as

will be specified below. First, by applying the Lemmas 3.1, 3.4 and Young inequality, we find that  $I_0$  can be bounded as follows

$$\begin{aligned} I_0 &\leq 2\mu_r \left| \int_{\Omega} (\operatorname{curl} \mathbf{w} \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (\operatorname{curl} \mathbf{u} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right| \\ &\leq 2\mu_r \left( \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} + \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right) \\ &\leq (\alpha)^{-1/2} \mu_r \Theta(t) \left( \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2 \right), \end{aligned} \quad (3.33)$$

for all  $t \in [0, T_1]$ . Now, by Lemmas 3.1 and 3.2, we get that

$$\begin{aligned} I_1 &\leq \left| \int_{\Omega} f'(t) (\rho(\nabla h - \mathbf{m}) \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} g'(t) (\rho \mathbf{q} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right| \\ &\leq \sqrt{\beta} \left( |f'(t)| \|\nabla h - \mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} + |g'(t)| \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right) \\ &\leq \sqrt{\beta} (\Pi_1 + 1) \left( |f'(t)| \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} + |g'(t)| \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2} \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right) \\ &\leq \overline{\Phi}_1 \left( |f'(t)| \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} + |g'(t)| \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right), \end{aligned} \quad (3.34)$$

where  $\overline{\Phi}_1 = \sqrt{\beta} (\Pi_1 + 1) \max\{\|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2}, \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2}\}$ . In the case of  $I_2$ , by applying Lemma 3.1, we have that

$$\begin{aligned} I_2 &\leq \left| \int_{\Omega} (f \rho(\nabla h_t - \mathbf{m}_t) \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (g \rho \mathbf{q}_t \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right| \\ &\leq \sqrt{\beta} \left( |f(t)| \left\{ \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} + \|\mathbf{m}_t(\cdot, t)\|_{\mathbf{L}^2} \right\} \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} \right. \\ &\quad \left. + |g(t)| \|\mathbf{q}_t(\cdot, t)\|_{\mathbf{L}^2} \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right) \\ &\leq \overline{\Phi}_2 \left( |f(t)| \left\{ \|\nabla h_t(\cdot, t)\|_{\mathbf{L}^2} + 1 \right\} \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} \right. \\ &\quad \left. + |g(t)| \left\{ \|\nabla r_t(\cdot, t)\|_{\mathbf{L}^2} + 1 \right\} \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right), \end{aligned} \quad (3.35)$$

where  $\overline{\Phi}_2 = \sqrt{\beta} \max\{\|\mathbf{m}_t(\cdot, t)\|_{\mathbf{L}^2}, \|\mathbf{q}_t(\cdot, t)\|_{\mathbf{L}^2}, 1\}$ . For  $I_3$ , by equation (1.4), inequality (3.8), Lemmas 3.1 and 3.4 and noticing that

$$\|(\nabla h - \mathbf{m})(\cdot, t)\|_{\mathbf{L}^3} \leq 2^{-1} \left( C_{gn}^2 \Pi_1^2 \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \|h(\cdot, t)\|_{H^2} + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^3}^2 \right)$$

we deduce that

$$\begin{aligned} I_3 &\leq \left| \int_{\Omega} f(t) (\rho_t(\nabla h - \mathbf{m}) \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} g(t) (\rho_t \mathbf{q} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right| \\ &= \left| \int_{\Omega} f(t) ((\mathbf{u} \cdot \nabla \rho)(\nabla h - \mathbf{m}) \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} g(t) ((\mathbf{w} \cdot \nabla \rho) \mathbf{q} \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right| \\ &\leq |f(t)| \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^3} \|(\nabla h - \mathbf{m})(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^6} \\ &\quad + |g(t)| \|\mathbf{w}(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^6} \\ &\leq C_{poi} \Theta(t) \left\{ |f(t)| \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^3} \|(\nabla h - \mathbf{m})(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^6} \right. \\ &\quad \left. + |g(t)| \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^6} \right\} \\ &\leq 2^{-1} C_{poi}^2 \Theta(t) \left\{ |f(t)| \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^3} \left( C_{gn}^2 \Pi_1^2 \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2} \|h(\cdot, t)\|_{H^2} + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^3}^2 \right) \times \right. \\ &\quad \left. \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} + |g(t)| \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^3}^2 \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \right\} \\ &\leq \frac{1}{2\epsilon_u} \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \overline{\Phi}_3^u |f(t)|^2 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^3}^2 \left( \|h(\cdot, t)\|_{H^2}^2 + 1 \right) \\ &\quad + \frac{1}{2\epsilon_w} \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \overline{\Phi}_3^w |g(t)|^2 \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^3}^2, \end{aligned} \quad (3.36)$$

for all  $t \in [0, T_1]$ , where  $\overline{\Phi}_3^\ell = 3\epsilon_\ell |\Omega|^{2(q-3)/3q} C_{poi}^4 \Theta(t)^2 \mathcal{L}^2 / 8$  with  $\ell \in \{\mathbf{u}, \mathbf{w}\}$  and  $\mathcal{L}$  is defined as follows

$$\mathcal{L} = \max \left\{ 2^{-1} C_{gn}^2 \Pi_1^2 \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^3}^2, 2^{-1} C_{gn}^2 \Pi_1^2 \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^2}^2 + \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^3}^2 \right\}.$$

The term  $I_4$  can be bounded by the application of equation (1.4), the inequality (3.8) and Lemma 3.4, since we can perform the following calculus

$$\begin{aligned} I_4 &\leq \frac{1}{2} \left| \int_{\Omega} (\rho_t |\mathbf{u}_t|^2)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (\rho_t |\mathbf{w}_t|^2)(\mathbf{x}, t) d\mathbf{x} \right| \\ &= \frac{1}{2} \left| \int_{\Omega} ((\mathbf{u} \cdot \nabla \rho) |\mathbf{u}_t|^2)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} ((\mathbf{w} \cdot \nabla \rho) |\mathbf{w}_t|^2)(\mathbf{x}, t) d\mathbf{x} \right| \\ &\leq \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^3} \\ &\quad + \|\mathbf{w}(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^3} \\ &\leq C_{poi} \Theta(t) \left\{ \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^3} \right. \\ &\quad \left. + \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^3} \right\} \\ &\leq C_{poi} C_{gn}^2 \Theta(t) \left\{ \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^{1-3/q} \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^{3/q} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^{1/2} \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^{1/2} \right. \\ &\quad \left. + \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^{1-3/q} \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^{3/q} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^{1/2} \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^{1/2} \right\} \\ &\leq \frac{1}{2\epsilon_{\mathbf{u}}} \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \overline{\Phi}_4^{\mathbf{u}} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^{4q/(3q-6)} \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{H}^2}^2 \\ &\quad + \frac{1}{2\epsilon_{\mathbf{w}}} \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \overline{\Phi}_4^{\mathbf{w}} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^{4q/(q+6)} \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{H}^2}^2, \end{aligned} \quad (3.37)$$

where  $\overline{\Phi}_4^\ell = (\alpha)^{-1} \left( (2\epsilon_\ell)^{(q+6)} (C_{poi} C_{gn}^2 \Theta(t))^{4q} \right)^{1/(3q-6)}$  with  $\ell \in \{\mathbf{u}, \mathbf{w}\}$ . An application of equation (1.4), inequality (3.8) and Lemma 3.4 implies the following bound for  $I_5$

$$\begin{aligned} I_5 &\leq \left| \int_{\Omega} (\rho_t (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{w}_t(\mathbf{x}, t) d\mathbf{x} \right| \\ &= \frac{1}{2} \left| \int_{\Omega} ((\mathbf{u} \cdot \nabla \rho) \mathbf{u}_t \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} ((\mathbf{w} \cdot \nabla \rho) \mathbf{w}_t \cdot \mathbf{w}_t)(\mathbf{x}, t) d\mathbf{x} \right| \\ &\leq \|\mathbf{u}(\cdot, t)\|_{\mathbf{L}^6}^2 \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \\ &\quad + \|\mathbf{w}(\cdot, t)\|_{\mathbf{L}^6}^2 \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^6} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^{2q/(q-2)}} \\ &\leq C_{poi}^3 \Theta(t)^2 C_{gn} \left\{ \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2}^{1-3/q} \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2}^{3/q} \right. \\ &\quad \left. + \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q} \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2}^{1-3/q} \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2}^{3/q} \right\} \\ &\leq \frac{1}{2\epsilon_{\mathbf{u}}} \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \overline{\Phi}_5^{\mathbf{u}} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^2 \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2}^{6/q} \\ &\quad + \frac{1}{2\epsilon_{\mathbf{w}}} \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \overline{\Phi}_5^{\mathbf{w}} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^2 \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2}^{6/q}, \end{aligned} \quad (3.38)$$

where  $\overline{\Phi}_5^\ell = 2^{-1} \epsilon_\ell C_{poi}^6 \Theta(t)^{6(q-1)/q} C_{gn}^2$  with  $\ell \in \{\mathbf{u}, \mathbf{w}\}$ . By inequality (3.8) and Lemma 3.4 we deduce that

$$\begin{aligned} I_6 &\leq \left| \int_{\Omega} (\rho (\mathbf{u}_t \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_t)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (\rho (\mathbf{w}_t \cdot \nabla) \mathbf{w} \cdot \mathbf{w}_t) d\mathbf{x} \right| \\ &\leq \|\rho(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^6} \\ &\quad + \|\rho(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^3} \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^6} \\ &\leq \beta C_{poi} C_{gn} \Theta(t) \left\{ \|\mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^{1/2} \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^{3/2} + \|\mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^{1/2} \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^{3/2} \right\} \end{aligned}$$

$$\leq \frac{1}{2\epsilon_{\mathbf{u}}} \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \overline{\Phi}_6^{\mathbf{u}} \|\sqrt{\rho} \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \frac{1}{2\epsilon_{\mathbf{w}}} \|\nabla \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2 + \overline{\Phi}_6^{\mathbf{w}} \|\sqrt{\rho} \mathbf{w}_t(\cdot, t)\|_{\mathbf{L}^2}^2, \quad (3.39)$$

where  $\overline{\Phi}_6^\ell = (\alpha)^{-1} (2\epsilon_\ell)^3 (\beta C_{poi} C_{gn} \Theta(t))^4$  with  $\ell \in \{\mathbf{u}, \mathbf{w}\}$ . Inserting (3.34)-(3.39) in (3.32) and selecting  $\epsilon_{\mathbf{u}} = 2(\mu + \mu_r)^{-1}$  and  $\epsilon_{\mathbf{w}} = 2(c_a + c_d)^{-1}$ , we deduce that (3.31) holds with  $\Phi_i = 2 \max\{\overline{\Phi}_i^{\mathbf{u}}, \overline{\Phi}_i^{\mathbf{w}}\}$  for  $i = 1, \dots, 6$ .  $\square$

**3.2. Proof of Theorem 3.1.** The existence of  $T_1$  and  $\kappa_1$  follows from (3.30). Now, before starting the proof of (3.2)-(3.7), we deduce two estimates. First, differentiating (1.4) with respect to  $x_i$ , using (1.2), testing the result by  $|\rho_{x_i}|^{q-2} \rho_{x_i}$  and applying the Sobolev inequality we deduce that there exists  $C_{sob}$  independent of  $f$  and  $g$  such that

$$\frac{d}{dt} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}^q \leq C_{sob} \|\mathbf{u}(\cdot, t)\|_{\mathbf{W}^{2,s}} \|\nabla \rho(\cdot, t)\|_{\mathbf{L}^q}, \quad \text{for } t \in [0, T_*]. \quad (3.40)$$

Second, by the regularity of the solutions for (3.19) we have that there exists  $C_3^{reg}$  depending only on  $\mu, \mu_r$  and  $\Omega$  such that

$$\|\mathbf{u}(\cdot, t)\|_{\mathbf{W}^{2,s}} + \|p(\cdot, t)\|_{\mathbf{W}^{1,s}} \leq C_3^{reg} \left\| \left( 2\mu_r \operatorname{curl} \mathbf{w} + \rho f(\nabla h - \mathbf{m}) - \rho \mathbf{u}_t - \rho(\mathbf{u} \cdot \nabla) \mathbf{u} \right) (\cdot, t) \right\|_{\mathbf{L}^s}.$$

Hence, by the Minkowski and Hölder inequalities and (3.8), we find that there exists  $\xi_1 = 2\mu_r C_3^{reg} C_{gn}$ ,  $\xi_2 = \beta C_3^{reg} \max\{C_{gn}, \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^2}\}$ ,  $\xi_3 = C_3^{reg} C_{gn} C_{iny}^{2,\infty}$  and  $\xi_4 = \beta C_3^{reg} C_{poi}$ , such that

$$\begin{aligned} \|\mathbf{u}(\cdot, t)\|_{\mathbf{W}^{2,s}} + \|p(\cdot, t)\|_{\mathbf{W}^{1,s}} &\leq \xi_1 \|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2} + \xi_2 |f(t)| \left( \|h(\cdot, t)\|_{H^2} + 1 \right) \\ &\quad + \xi_3 \|\mathbf{u}(\cdot, t)\|_{\mathbf{H}^2}^2 + \xi_4 \|\nabla \mathbf{u}_t(\cdot, t)\|_{\mathbf{L}^2}, \end{aligned} \quad (3.41)$$

for  $t \in [0, T_*]$ . Therefore, we derive the proof of (3.2) by inserting (3.41) in (3.40) and using the estimates (3.15), (3.16) and (3.31). The estimate (3.3) is deduced from (3.2) and (3.15). The inequality (3.4) is obtained from (3.2), (3.12), (3.13), (3.16) and (3.18). The estimate (3.5) is proved by the application of (3.1), (3.12), (3.13) and (3.18). The estimate (3.6) follows from (1.4), (3.1) and (3.5). We complete the proof of the theorem deducing the inequality (3.7) by combining the results given on (3.2) and (3.41).

#### 4. WELL-POSEDNESS OF THE DIRECT PROBLEM.

The well-posedness of the direct problem is given by the following Theorem:

**Theorem 4.1.** *Consider that the functions  $\rho_0, \mathbf{v}_0, \mathbf{w}_0, \mathbf{m}, \mathbf{q}, h$  and  $r$  satisfy the hypothesis  $(H_1)$ -( $H_4$ ) and  $(f, g) \in [H^1(0, T)]^2$ . Then, the direct problem (1.1)-(1.5) and (1.8)-(1.11) possesses a unique solution  $\{\mathbf{u}, \mathbf{w}, \rho, p, h\}$  in the sense of the definition 2.1.*

We note the proof of existence of solutions can be devolving by applying the ideas of Boldrini et al. [6]. Meanwhile, the uniqueness is a straightforward consequence of a continuous dependence of the system unknowns with respect to the source coefficients, which is proved in the following Lemma.

**Lemma 4.1.** *Consider that the functions  $\rho_0, \mathbf{v}_0, \mathbf{w}_0$  and  $h$  satisfying the hypothesis  $(H_1)$ -( $H_4$ ). Suppose that  $\{\mathbf{u}_i, \mathbf{w}_i, \rho_i, p_i, h_i\}$ ,  $i = 1, 2$ , are two solutions of the direct problem (1.1)-(1.5) and (1.8)-(1.11) corresponding to the coefficients  $(f_i, g_i) \in [H^1(0, T)]^2$ ,  $i = 1, 2$ , respectively. There exists  $C$  independent of  $(f_i, g_i)$ ,  $i = 1, 2$ , such that the following estimate hold:*

$$\begin{aligned} &\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty([0,t]; \mathbf{L}^2(\Omega))} + \|\mathbf{w}_1 - \mathbf{w}_2\|_{L^\infty([0,t]; \mathbf{L}^2(\Omega))} \\ &\quad + \|\rho_1 - \rho_2\|_{L^\infty([0,t]; L^2(\Omega))} + \|\nabla(h_1 - h_2)\|_{L^\infty([0,t]; \mathbf{L}^2(\Omega))} \\ &\leq C \left( \|f_1 - f_2\|_{L^2([0,t])} + \|g_1 - g_2\|_{L^2([0,t])} \right) \end{aligned} \quad (4.1)$$

for all  $t$  belongs to the maximal interval where the solutions are defined.

*Proof.* In order to simplify the presentation of the estimates we introduce the following notation

$$\begin{aligned}\delta \mathbf{u} &= \mathbf{u}_1 - \mathbf{u}_2, & \delta \rho &= \rho_1 - \rho_2, & \delta h &= h_1 - h_2, & \delta \mathbf{w} &= \mathbf{w}_1 - \mathbf{w}_2, \\ \delta f &= f_1 - f_2, & \delta g &= g_1 - g_2, & \delta p &= p_1 - p_2 & \text{ and } & \bar{s} &= 2s/(s-2),\end{aligned}$$

where  $s \in [2, \infty[$ . Hence, by algebraic rearrangements of both forward problems ( $i = 1$  and  $i = 2$ ), we get the following system

$$\begin{aligned}\rho_1 \delta \mathbf{u}_t + \rho_1 (\mathbf{u}_1 \cdot \nabla) \delta \mathbf{u} - (\mu + \mu_r) \Delta \delta \mathbf{u} + \nabla \delta p &= -\delta \rho (\mathbf{u}_2)_t + [(\delta \rho \mathbf{u}_1 + \rho_2 \delta \mathbf{u}) \cdot \nabla] \mathbf{u}_2 \\ &+ 2\mu_r \operatorname{curl} \delta \mathbf{w} + \delta \rho f_1 \nabla h_1 + \rho_2 \delta f \nabla h_1 \\ &+ \rho_2 f_2 \nabla \delta h - (\delta \rho f_1 + \rho_2 \delta f) \mathbf{m}, \quad \text{in } Q_T,\end{aligned}\tag{4.2}$$

$$\operatorname{div} \delta \mathbf{u} = 0, \quad \text{in } Q_T,\tag{4.3}$$

$$\begin{aligned}\rho_1 \delta \mathbf{w}_t + \rho_1 (\mathbf{w}_1 \cdot \nabla) \delta \mathbf{w} - (c_a + c_d) \Delta \delta \mathbf{w} &= -\delta \rho (\mathbf{w}_2)_t + [(\delta \rho \mathbf{w}_1 + \rho_2 \delta \mathbf{w}) \cdot \nabla] \mathbf{w}_2 - 4\mu_r \delta \mathbf{w} \\ &+ (c_o + c_d - c_a) \nabla \operatorname{div} \delta \mathbf{w} + 2\mu_r \operatorname{curl} \delta \mathbf{u} + (\delta \rho g_1 + \rho_2 \delta g) \mathbf{q}, \quad \text{in } Q_T,\end{aligned}\tag{4.4}$$

$$\delta \rho_t + \delta \mathbf{u} \cdot \nabla \rho_1 + \mathbf{u}_2 \cdot \nabla \delta \rho = 0, \quad \text{in } Q_T,\tag{4.5}$$

$$\delta \mathbf{u}(\mathbf{x}, t) = \delta \mathbf{w}(\mathbf{x}, t) = 0, \quad \text{on } \Sigma_T,\tag{4.6}$$

$$\delta \mathbf{u}(\mathbf{x}, 0) = \delta \mathbf{w}(\mathbf{x}, 0) = \delta \rho(\mathbf{x}, 0) = 0, \quad \text{on } \Omega,\tag{4.7}$$

$$\operatorname{div} (\rho_1 \nabla \delta h) = \operatorname{div} (\delta \rho (\mathbf{m} - \nabla h_2)), \quad \text{in } \Omega,\tag{4.8}$$

$$\frac{\partial \delta h}{\partial \mathbf{n}}(\mathbf{x}, t) = 0, \quad \text{on } \Sigma_T,\tag{4.9}$$

$$\int_{\Omega} \delta h(\mathbf{x}, t) d\mathbf{x} = 0, \quad t \in [0, T].\tag{4.10}$$

Now, to prove (4.1) we proceed in two big steps: first we obtain five a priori estimates for the system (4.2)-(4.10) and then we apply the Gronwall inequality.

First, by equations (1.2) for  $u_2$  and (4.5); the boundary condition (4.6); integration by parts; the Hölder and Young's inequalities; and (3.8) for  $p = 2$  and  $q = \bar{s} \in ]2, 6[$ , we have that

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \int_{\Omega} \delta \rho^2(\mathbf{x}, t) d\mathbf{x} &= \int_{\Omega} (\delta \rho_t \delta \rho)(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} -[\delta \mathbf{u} \cdot \nabla \rho_1 + \mathbf{u}_2 \cdot \nabla \delta \rho] \delta \rho(\mathbf{x}, t) d\mathbf{x} \\ &= - \int_{\Omega} \delta \rho (\delta \mathbf{u} \cdot \nabla \rho_1)(\mathbf{x}, t) d\mathbf{x} + \frac{1}{2} \int_{\Omega} \delta \rho^2 \operatorname{div} (\mathbf{u}_2)(\mathbf{x}, t) d\mathbf{x} \\ &\leq \| \nabla \rho_1(\cdot, t) \|_{L^s(\Omega)} \| \delta \mathbf{u}(\cdot, t) \|_{L^{\bar{s}}(\Omega)} \| \delta \rho(\cdot, t) \|_{L^2(\Omega)} \\ &\leq \frac{1}{2\epsilon_{\mathbf{u}}} \| \nabla \rho_1(\cdot, t) \|_{L^s(\Omega)}^2 \| \delta \rho(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{\epsilon_{\mathbf{u}}}{2} C_{poi} \| \nabla \delta \mathbf{u}(\cdot, t) \|_{L^2(\Omega)}^2,\end{aligned}\tag{4.11}$$

where  $\epsilon_{\mathbf{u}} > 0$  is the parameter used for the Young's inequality. In the second place, by equation (1.4) for  $\rho_1$  and the boundary condition (4.6), we note that the left hand side of (4.2) multiplied by  $\delta \mathbf{u}$  can be integrated by parts and simplified as follows

$$\begin{aligned}\int_{\Omega} [\rho_1 \delta \mathbf{u}_t + \rho_1 (\mathbf{u}_1 \cdot \nabla) \delta \mathbf{u} - (\mu + \mu_r) \Delta \delta \mathbf{u} + \nabla \delta p] \cdot \delta \mathbf{u}(\mathbf{x}, t) d\mathbf{x} \\ = \int_{\Omega} \left[ \frac{1}{2} \left\{ (\rho_1 |\delta \mathbf{u}|^2)_t - (\rho_1)_t |\delta \mathbf{u}|^2 \right\} + \rho_1 [(\mathbf{u}_1 \cdot \nabla) \delta \mathbf{u}] \cdot \delta \mathbf{u} \right. \\ \left. - (\mu + \mu_r) \Delta \delta \mathbf{u} \cdot \delta \mathbf{u} + \nabla \delta p \cdot \delta \mathbf{u} \right](\mathbf{x}, t) d\mathbf{x} \\ = \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\rho_1 |\delta \mathbf{u}|^2)(\mathbf{x}, t) d\mathbf{x} + (\mu + \mu_r) \int_{\Omega} |\nabla \delta \mathbf{u}|^2(\mathbf{x}, t) d\mathbf{x}.\end{aligned}\tag{4.12}$$

Then, a multiplication of (4.2) by  $\delta \mathbf{u}$ , integration by parts and application of the Hölder inequality leads to

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\rho_1 |\delta \mathbf{u}|^2)(\mathbf{x}, t) d\mathbf{x} + (\mu + \mu_r) \int_{\Omega} |\nabla \delta \mathbf{u}|^2(\mathbf{x}, t) d\mathbf{x}$$

$$\begin{aligned}
&\leq \|(\mathbf{u}_2)_t(\cdot, t)\|_{\mathbf{L}^s} \|\delta\rho(\cdot, t)\|_{\mathbf{L}^2} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^s} + \|\mathbf{u}_1(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla\mathbf{u}_2(\cdot, t)\|_{\mathbf{L}^s} \|\delta\rho(\cdot, t)\|_{\mathbf{L}^2} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^s} \\
&+ \|\rho_2(\cdot, t)\|_{\mathbf{L}^\infty} \|\nabla\mathbf{u}_2(\cdot, t)\|_{\mathbf{L}^s} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^s} + 2\mu_r \|\nabla\delta\mathbf{w}(\cdot, t)\|_{\mathbf{L}^2} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \\
&+ |f_1(t)| \|\nabla h_1(\cdot, t)\|_{\mathbf{L}^s} \|\delta\rho(\cdot, t)\|_{\mathbf{L}^2} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^s} \\
&+ |\delta f(t)| \|\nabla h_1(\cdot, t)\|_{\mathbf{L}^s} \|\rho_2(\cdot, t)\|_{\mathbf{L}^\infty} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} |\Omega|^{1/\bar{s}} \\
&+ |f_2(t)| \|\nabla\delta h(\cdot, t)\|_{\mathbf{L}^2} \|\rho_2(\cdot, t)\|_{\mathbf{L}^\infty} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} + |f_1(t)| \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty} \|\delta\rho(\cdot, t)\|_{\mathbf{L}^2} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} \\
&+ |\delta f(t)| \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty} \|\rho_2(\cdot, t)\|_{\mathbf{L}^\infty} \|\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2} |\Omega|^{1/2}. \tag{4.13}
\end{aligned}$$

Hence by the Young's inequality and (3.8), we have that there exists  $\Gamma_i^{\mathbf{u}} := \Gamma_i^{\mathbf{u}}(t) > 0$ ,  $i \in \{1, 2, 3\}$ , such that (see A for details)

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\rho_1 |\delta\mathbf{u}|^2)(\mathbf{x}, t) d\mathbf{x} + (\mu + \mu_r) \int_{\Omega} |\nabla\delta\mathbf{u}|^2(\mathbf{x}, t) d\mathbf{x} \\
&\leq \Gamma_1^{\mathbf{u}} \left( \|\sqrt{\rho_1} \delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2 + \|\delta\rho(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2 \right) + \Gamma_2^{\mathbf{u}} \|\nabla\delta h(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2 + \Gamma_3^{\mathbf{u}} |\delta f(t)|^2 \\
&+ 2\epsilon_{\mathbf{u}} C_{poi} \|\nabla\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\epsilon_{\mathbf{w}}}{2} \|\nabla\delta\mathbf{w}(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2. \tag{4.14}
\end{aligned}$$

For the third estimate, we start from (4.4) and proceeding by a similar reasoning to the steps (4.12)-(4.14) we deduce that there exists  $\Gamma_i^{\mathbf{w}} := \Gamma_i^{\mathbf{w}}(t) > 0$ ,  $i \in \{1, 2\}$ , such that (see A)

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\rho_1 |\delta\mathbf{w}|^2)(\mathbf{x}, t) d\mathbf{x} + (c_a + c_d) \int_{\Omega} |\nabla\delta\mathbf{w}|^2(\mathbf{x}, t) d\mathbf{x} \\
&+ (c_0 + c_d - c_a) \int_{\Omega} |\operatorname{div} \delta\mathbf{w}|^2(\mathbf{x}, t) d\mathbf{x} + 4u_r \int_{\Omega} |\delta\mathbf{w}|^2(\mathbf{x}, t) d\mathbf{x} \\
&\leq \Gamma_1^{\mathbf{w}} \left( \|\sqrt{\rho_1} \delta\mathbf{w}(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2 + \|\delta\rho(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2 \right) + \Gamma_2^{\mathbf{w}} |\delta g(t)|^2 \\
&+ \frac{\epsilon_{\mathbf{u}}}{2} \|\nabla\delta\mathbf{u}(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2 + 2\epsilon_{\mathbf{w}} C_{poi} \|\nabla\delta\mathbf{w}(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2. \tag{4.15}
\end{aligned}$$

In the fourth place, we deduce an estimate related to  $\delta h$ . Indeed, by equations for  $\delta h$  given on (4.8)-(4.10), integration by parts, the Hölder and Minkowski inequalities, and Lemma 3.1, we deduce that

$$\begin{aligned}
\alpha \|\nabla\delta h(\cdot, t)\|_{\mathbf{L}^2(\Omega)}^2 &\leq \int_{\Omega} \left( \rho_1 |\nabla\delta h|^2 \right)(\mathbf{x}, t) d\mathbf{x} = - \int_{\Omega} \operatorname{div} (\rho_1 \nabla\delta h) \delta h(\mathbf{x}, t) d\mathbf{x} \\
&= - \int_{\Omega} \left[ \operatorname{div} (\delta\rho(\mathbf{m} - \nabla h_2)) \delta h \right](\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} \left[ \delta\rho(\mathbf{m} - \nabla h_2) \nabla\delta h \right](\mathbf{x}, t) d\mathbf{x} \\
&\leq \|\delta\rho(\mathbf{m} - \nabla h_2)(\cdot, t)\|_{\mathbf{L}^2(\Omega)} \|\nabla\delta h(\cdot, t)\|_{\mathbf{L}^2(\Omega)} \\
&\leq \left( \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla h_2(\cdot, t)\|_{\mathbf{L}^\infty(\Omega)} \right) \|\delta\rho(\cdot, t)\|_{\mathbf{L}^2(\Omega)} \|\nabla\delta h(\cdot, t)\|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Thus, we have the following two estimates

$$\|\nabla\delta h(\cdot, t)\|_{\mathbf{L}^2(\Omega)} \leq \frac{1}{\alpha} \left( \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla h_2(\cdot, t)\|_{\mathbf{L}^\infty(\Omega)} \right) \|\delta\rho(\cdot, t)\|_{\mathbf{L}^2(\Omega)}. \tag{4.17}$$

From (4.11) and (4.14), selecting  $\epsilon_{\mathbf{u}} = 2(\mu + \mu_r)(5C_{poi} + 1)^{-1}$  and  $\epsilon_{\mathbf{w}} = 2(c_a + c_d)(5C_{poi} + 1)^{-1}$ , we deduce that

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega} (\rho_1 |\delta\mathbf{u}|^2 + \rho_1 |\delta\mathbf{w}|^2 + \delta\rho^2)(\mathbf{x}, t) d\mathbf{x} + \int_{\Omega} (\nabla\delta h)^2(\mathbf{x}, t) d\mathbf{x} \\
&\leq \Gamma^\delta(t) \int_{\Omega} (\rho_1 |\delta\mathbf{u}|^2 + \rho_1 |\delta\mathbf{w}|^2 + \delta\rho^2)(\mathbf{x}, t) d\mathbf{x} + \Gamma_3^{\mathbf{u}}(t) |\delta f(t)|^2 + \Gamma_3^{\mathbf{w}}(t) |\delta g(t)|^2.
\end{aligned}$$

where  $\Gamma^\delta(t) = \max\{\Gamma_1^{\mathbf{u}}(t), \Gamma_1^{\mathbf{w}}(t), C^\delta(t)\}$ , with  $C^\delta$  defined as follows

$$\begin{aligned}
C^\delta(t) &= \frac{1}{2\epsilon_{\mathbf{u}}} \|\nabla\rho_1(\cdot, t)\|_{\mathbf{L}^2(\Omega)} + \Gamma_1^{\mathbf{u}}(t) + \frac{\Gamma_2^{\mathbf{u}}(t)}{\alpha^2} \left( \|\mathbf{m}(\cdot, t)\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla h_2(\cdot, t)\|_{\mathbf{L}^\infty(\Omega)} \right)^2 \\
&+ \Gamma_1^{\mathbf{w}}(t) + \frac{\Gamma_2^{\mathbf{w}}(t)}{\alpha^2} \left( \|\mathbf{q}(\cdot, t)\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla g_2(\cdot, t)\|_{\mathbf{L}^\infty(\Omega)} \right)^2.
\end{aligned}$$



We note that  $\|\Gamma^\delta\|_{L^1(0,T)} < \infty$  and  $\Gamma_3^u(t) < \infty$ . Hence, we complete the proof of (4.1) by application of the Gronwall inequality.  $\square$

## 5. WELL-POSEDNESS OF THE INVERSE PROBLEM

The main result of this section is the derivation of a well-posedness result for the inverse problem, which is formalized by the following theorem:

**Theorem 5.1.** *Assume that  $(H_1)$ – $(H_7)$  are satisfied. Then there exists a unique solution  $\{\mathbf{u}, \mathbf{w}, \rho, p, h, f, g\}$  of the inverse problem defined on a small enough time  $T_*$ .*

The proof of Theorem 5.1 is detailed on subsection 5.3. It is based on a reformulation of the inverse problem like as an operator equation of second kind and then by application of the fixed point argument.

**5.1. Formulation of the inverse problem (1.1)–(1.11) as and operator equation.** We define the nonlinear operator

$$\begin{aligned} \mathcal{R} &: [H^1(0, T)]^2 \rightarrow [H^1(0, T)]^2 \\ (f, g) &\mapsto (\mathcal{R}_1, \mathcal{R}_2) := \left( \frac{\mathcal{N}_1}{\gamma_1}, \frac{\mathcal{N}_2}{\gamma_2} \right) \end{aligned} \quad (5.1)$$

where

$$\mathcal{N}_1(t) = \frac{d\phi^{\mathbf{u}}}{dt}(t) - \int_{\Omega} \rho \mathbf{u} \otimes \mathbf{u} : \nabla \psi^{\mathbf{u}} dx + (\mu + \mu_r) (A\mathbf{u}, \psi^{\mathbf{u}}) - 2\mu_r (\text{curl } \mathbf{w}, \psi^{\mathbf{u}}), \quad (5.2)$$

$$\mathcal{N}_2(t) = \frac{d\phi^{\mathbf{w}}}{dt}(t) - \int_{\Omega} \rho \mathbf{w} \otimes \mathbf{w} : \nabla \psi^{\mathbf{w}} dx + (L\mathbf{w}, \psi^{\mathbf{w}}) - 2\mu_r (\text{curl } \mathbf{u}, \psi^{\mathbf{w}}), \quad (5.3)$$

$$\gamma_1(t) = \int_{\Omega} \rho(x, t) (\nabla h(x, t) - \mathbf{m}(x, t)) \cdot \psi^{\mathbf{u}}(x) dx \quad \text{and} \quad (5.4)$$

$$\gamma_2(t) = \int_{\Omega} \rho(x, t) \mathbf{q}(x, t) \cdot \psi^{\mathbf{w}}(x) dx. \quad (5.5)$$

We note that the solvability of the inverse problem (1.1)–(1.11) is connected with the following operator equation of the second kind

$$\mathcal{R}(f, g) = (f, g) \quad \text{over} \quad D := \{(f, g) \in [H^1(0, T)]^2 : \|(f, g)\|_{[H^1(0, T)]^2} \leq \eta\}, \quad (5.6)$$

where  $\eta \in \mathbb{R}^+$  will be constructed in order to get the solvability of (5.6) by application of the fixed point argument. More specifically, we have the following characterization result of the solvability of the inverse problem.

**Theorem 5.2.** *Let  $(H_1)$ – $(H_7)$  be satisfied. Then the following two assertions are valid:*

- (i) *If the inverse problem (1.1)–(1.11) is solvable, then so is equation (5.6), and*
- (ii) *If (5.6) is solvable and the following compatibility conditions*

$$\int_{\Omega} \rho(\mathbf{x}, 0) \mathbf{u}(\mathbf{x}, 0) \cdot \psi^{\mathbf{u}}(\mathbf{x}) d\mathbf{x} = \phi^{\mathbf{u}}(0) \quad \text{and} \quad \int_{\Omega} \rho(\mathbf{x}, 0) \mathbf{w}(\mathbf{x}, 0) \cdot \psi^{\mathbf{w}}(\mathbf{x}) d\mathbf{x} = \phi^{\mathbf{w}}(0) \quad (5.7)$$

*are satisfied, then there exists a solution of the inverse problem (1.1)–(1.11).*

*Proof.* (i) Let us consider  $\{\mathbf{u}, \mathbf{w}, \rho, p, h, f, g\}$  a solution of the inverse problem (1.1)–(1.11). By the overdetermination condition (1.7) and the definitions of  $\gamma_1$  and  $\gamma_2$ , given on (5.4) and (5.5), we deduce the following identities

$$\begin{aligned} ((\rho \mathbf{u})_t, \psi^{\mathbf{v}}) &= \frac{d\phi^{\mathbf{u}}}{dt}(t), \quad ((\rho \mathbf{w})_t, \psi^{\mathbf{w}}) = \frac{d\phi^{\mathbf{w}}}{dt}(t), \\ (\rho \mathbf{F}, \psi^{\mathbf{v}}) &= f(t) \gamma_1(t) \quad \text{and} \quad (\rho \mathbf{G}, \psi^{\mathbf{w}}) = g(t) \gamma_2(t). \end{aligned}$$

Hence, the selection  $\mathbf{v} = \psi^{\mathbf{u}}$  and  $\varphi = \psi^{\mathbf{w}}$  on the integral identities (2.6)–(2.7) leads to

$$\frac{d\phi^{\mathbf{u}}}{dt}(t) - \int_{\Omega} \rho \mathbf{u} \otimes \mathbf{u} : \nabla \psi^{\mathbf{u}} d\mathbf{x} + (\mu + \mu_r) (A\mathbf{u}, \psi^{\mathbf{u}}) - 2\mu_r (\text{curl } \mathbf{w}, \psi^{\mathbf{u}}) = f(t) \gamma_1(t), \quad (5.8)$$

$$\frac{d\phi^{\mathbf{w}}}{dt}(t) - \int_{\Omega} \rho \mathbf{w} \otimes \mathbf{w} : \nabla \psi^{\mathbf{w}} d\mathbf{x} + (L\mathbf{w}, \psi^{\mathbf{w}}) - 2\mu_r (\text{curl } \mathbf{u}, \psi^{\mathbf{w}}) = g(t)\gamma_2(t). \quad (5.9)$$

Clearly, in view of (5.8)-(5.9), we conclude that  $(f, g)$  solves the operator equation (5.6).

(ii). Let us consider  $(f, g) \in [H^1(0, T)]^2$  a solution of the operator equation (5.6). Then the relations (5.8) and (5.9) are satisfied and there exists  $\{\mathbf{u}, \mathbf{w}, \rho, p, h\}$  a solution of the direct problem (1.1)-(1.5) and (1.8)-(1.11). By selecting  $\mathbf{v} = \psi^{\mathbf{u}}$  and  $\varphi = \psi^{\mathbf{w}}$  on the integral identities (2.6)-(2.7), we arrive at

$$((\rho \mathbf{u})_t, \psi^{\mathbf{u}}) - \int_{\Omega} \rho \mathbf{u} \otimes \mathbf{u} : \nabla \psi^{\mathbf{u}} d\mathbf{x} + (\mu + \mu_r)(A\mathbf{u}, \psi^{\mathbf{u}}) - 2\mu_r (\text{curl } \mathbf{w}, \psi^{\mathbf{u}}) = f(t)\gamma_1(t), \quad (5.10)$$

$$((\rho \mathbf{w})_t, \psi^{\mathbf{w}}) - \int_{\Omega} \rho \mathbf{w} \otimes \mathbf{w} : \nabla \psi^{\mathbf{w}} d\mathbf{x} + (L\mathbf{w}, \psi^{\mathbf{w}}) - 2\mu_r (\text{curl } \mathbf{u}, \psi^{\mathbf{w}}) = g(t)\gamma_2(t). \quad (5.11)$$

Subtracting, (5.10) from (5.8) and (5.11) from (5.9) we deduce that

$$((\rho \mathbf{u})_t, \psi^{\mathbf{u}}) - \frac{d\phi^{\mathbf{u}}}{dt}(t) = 0 \quad \text{and} \quad ((\rho \mathbf{w})_t, \psi^{\mathbf{w}}) - \frac{d\phi^{\mathbf{w}}}{dt}(t) = 0. \quad (5.12)$$

Integrating the equations (5.12) and using the compatibility conditions given on (5.7), we deduce that the overdetermination conditions (1.7) are satisfied. Thus, we have that  $\{\mathbf{u}, \mathbf{w}, \rho, p, h, f, g\}$  is a solution of the inverse problem (1.1)-(1.11).  $\square$

## 5.2. Properties of the operator $\mathcal{R}$ defined on (5.1)-(5.5).

**Lemma 5.1.** *Assume that  $(H_1)$ -( $H_7$ ) are satisfied. Then there exists a small enough time  $T_*$  and there exists  $\eta \in \mathbb{R}^+$  such that  $\mathcal{R}$  maps  $D$  into itself.*

*Proof.* From ( $H_7$ ) and Theorem 4.1 we deduce that there exists  $T_{\bowtie} \in [0, T_*]$  such that

$$\gamma_1, \gamma_2 \in C[0, T_{\bowtie}], \quad |\gamma_1(t)| \geq h^\epsilon > 0 \quad \text{and} \quad |\gamma_2(t)| \geq r^\epsilon > 0 \quad \text{on} \quad [0, T_{\bowtie}]. \quad (5.13)$$

Then, by the definition of  $\mathcal{R}$  given on (5.1), we have that

$$\begin{aligned} \|\mathcal{R}(f, g)\|_{[H^1([0, T])]^2}^2 &= \sum_{i=1}^2 \|\mathcal{R}_i(f, g)\|_{H^1([0, T])}^2 \\ &\leq \frac{1}{(h^\epsilon)^2} \|\mathcal{N}_1\|_{L^2([0, T])}^2 + \left( \frac{1}{(h^\epsilon)^2} \left\| \frac{d\mathcal{N}_1}{dt} \right\|_{L^2([0, T])}^2 + \frac{1}{(h^\epsilon)^4} \|\mathcal{N}_1\|_{L^2([0, T])} \left\| \frac{d\gamma_1}{dt} \right\|_{L^2([0, T])} \right)^2 \\ &\quad + \frac{1}{(r^\epsilon)^2} \|\mathcal{N}_2\|_{L^2([0, T])}^2 + \left( \frac{1}{(r^\epsilon)^2} \left\| \frac{d\mathcal{N}_2}{dt} \right\|_{L^2([0, T])}^2 + \frac{1}{(r^\epsilon)^4} \|\mathcal{N}_2\|_{L^2([0, T])} \left\| \frac{d\gamma_2}{dt} \right\|_{L^2([0, T])} \right)^2 \end{aligned} \quad (5.14)$$

Now, denoting by  $\|\cdot\|_{p, q}$  the norm  $\|\cdot\|_{L^p([0, T]; L^q(\Omega))}$  and by  $\|\cdot\|_{\mathbf{L}^p}$  the norm  $\|\cdot\|_{\mathbf{L}^p(\Omega)}$ , making use of the relation (2.1) for the operator  $A$  and applying the Minkowski and Hölder inequalities, we have that

$$\begin{aligned} \|\mathcal{N}_1\|_{L^2([0, T])} &\leq \left\| \frac{d\phi^{\mathbf{u}}}{dt} \right\|_{L^2([0, T])} + \|\rho\|_{\infty, \infty} \|\mathbf{u}\|_{\infty, 2}^2 \|\psi^{\mathbf{u}}\|_{\mathbf{L}^\infty} T^{1/2} + (\mu + \mu_r) \|\nabla \mathbf{u}\|_{\infty, 2} \times \\ &\quad \|\nabla \psi^{\mathbf{u}}\|_{\mathbf{L}^2} T^{1/2} + 2\mu_r \|\nabla \mathbf{w}\|_{\infty, 2} \|\psi^{\mathbf{u}}\|_{\mathbf{L}^2} T^{1/2}, \\ \|\mathcal{N}_2\|_{L^2([0, T])} &\leq \left\| \frac{d\phi^{\mathbf{w}}}{dt} \right\|_{L^2([0, T])} + \|\rho\|_{\infty, \infty} \|\mathbf{w}\|_{\infty, 2}^2 \|\psi^{\mathbf{w}}\|_{\mathbf{L}^\infty} T^{1/2} + \|\mathbf{w}\|_{\infty, 2} \|L\psi^{\mathbf{w}}\|_{\infty, 2} \\ &\quad + 2\mu_r \|\nabla \mathbf{w}\|_{\infty, 2} \|\psi^{\mathbf{w}}\|_{\mathbf{L}^2} T^{1/2}, \\ \left\| \frac{d\mathcal{N}_1}{dt} \right\|_{L^2([0, T])} &\leq \left\| \frac{d^2\phi^{\mathbf{u}}}{dt^2} \right\|_{L^2([0, T])} + \|\rho_t\|_{\infty, 2} \|\mathbf{u}\|_{\infty, 6}^2 \|\psi^{\mathbf{u}}\|_{\mathbf{L}^6} T^{1/2} + 2\|\rho\|_{\infty, \infty} \|\mathbf{u}_t\|_{\infty, 2} \|\mathbf{u}\|_{\infty, 6} \times \\ &\quad \|\psi^{\mathbf{u}}\|_{\mathbf{L}^3} T^{1/2} + (\mu + \mu_r) \|\nabla \mathbf{u}_t\|_{\infty, 2} \|\nabla \psi^{\mathbf{u}}\|_{\mathbf{L}^2} T^{1/2} + 2\mu_r \|\nabla \mathbf{w}_t\|_{\infty, 2} \|\psi^{\mathbf{u}}\|_{\mathbf{L}^2} T^{1/2}, \\ \left\| \frac{d\mathcal{N}_2}{dt} \right\|_{L^2([0, T])} &\leq \left\| \frac{d^2\phi^{\mathbf{w}}}{dt^2} \right\|_{L^2([0, T])} + \|\rho_t\|_{\infty, 2} \|\mathbf{w}\|_{\infty, 6}^2 \|\psi^{\mathbf{w}}\|_{\mathbf{L}^6} T^{1/2} + 2\|\rho\|_{\infty, \infty} \|\mathbf{w}_t\|_{\infty, 2} \|\mathbf{w}\|_{\infty, 6} \times \\ &\quad \|\psi^{\mathbf{w}}\|_{\mathbf{L}^3} T^{1/2} + \|\mathbf{w}_t\|_{\infty, 2} \|L\psi^{\mathbf{w}}\|_{\mathbf{L}^2} + 2\mu_r \|\nabla \mathbf{w}_t\|_{\infty, 2} \|\psi^{\mathbf{u}}\|_{\mathbf{L}^2} T^{1/2}, \end{aligned}$$

$$\left\| \frac{d\gamma_1}{dt} \right\|_{L^2([0,T])} \leq \|\rho_t\|_{\infty,2} \|\nabla h - \mathbf{m}\|_{\infty,2} \|\psi^{\mathbf{u}}\|_{\mathbf{L}^\infty} T^{1/2} + \|\rho\|_{\infty,\infty} \|\nabla h_t - \mathbf{m}_t\|_{\infty,2} \|\psi^{\mathbf{u}}\|_{\mathbf{L}^2} T^{1/2} \text{ and}$$

$$\left\| \frac{d\gamma_2}{dt} \right\|_{L^2([0,T])} \leq \|\rho_t\|_{\infty,2} \|\mathbf{q}\|_{\infty,2} \|\psi^{\mathbf{w}}\|_{\mathbf{L}^\infty} T^{1/2} + \|\rho\|_{\infty,\infty} \|\nabla \mathbf{q}_t\|_{\infty,2} \|\psi^{\mathbf{w}}\|_{\mathbf{L}^2} T^{1/2}.$$

Hence, by Theorem 3.1 and Lemmas 3.1-3.2, we have that

$$\|\mathcal{N}_i\|_{L^2([0,T])} \leq \eta_i, \quad \|d\mathcal{N}_i/dt\|_{L^2([0,T])} \leq \bar{\eta}_i \quad \text{and} \quad \|d\gamma_i/dt\|_{L^2([0,T])} \leq \sigma_i, \quad i = 1, 2,$$

where

$$\begin{aligned} \eta_1 &= \left\| \frac{d\phi^{\mathbf{u}}}{dt} \right\|_{L^2([0,T])} + \kappa_1 T^{1/2} \left( \beta \kappa_1 \|\psi^{\mathbf{u}}\|_{\mathbf{L}^\infty} + (\mu + \mu_r) \|\nabla \psi^{\mathbf{u}}\|_{\mathbf{L}^2} + 2\mu_r \|\psi^{\mathbf{u}}\|_{\mathbf{L}^2} \right), \\ \eta_2 &= \left\| \frac{d\phi^{\mathbf{w}}}{dt} \right\|_{L^2([0,T])} + \kappa_1 T^{1/2} \left( \beta \kappa_1 \|\psi^{\mathbf{w}}\|_{\mathbf{L}^\infty} + \|L\psi^{\mathbf{w}}\|_{\mathbf{L}^2} + 2\mu_r \|\psi^{\mathbf{w}}\|_{\mathbf{L}^2} \right), \\ \bar{\eta}_1 &= \left\| \frac{d^2\phi^{\mathbf{u}}}{dt^2} \right\|_{L^2([0,T])} + T^{1/2} \left( \kappa_8 (C_{poi} \kappa_1)^2 \|\psi^{\mathbf{u}}\|_{\mathbf{L}^6} + 2\beta \kappa_1 \kappa_2 C_{poi} \|\psi^{\mathbf{u}}\|_{\mathbf{L}^3} \right. \\ &\quad \left. + (\mu + \mu_r) \kappa_2 \|\nabla \psi^{\mathbf{u}}\|_{\mathbf{L}^2} + 2\mu_r \kappa_2 \|\psi^{\mathbf{u}}\|_{\mathbf{L}^2} \right), \\ \bar{\eta}_2 &= \left\| \frac{d^2\phi^{\mathbf{w}}}{dt^2} \right\|_{L^2([0,T])} + T^{1/2} \left( \kappa_8 (C_{poi} \kappa_1)^2 \|\psi^{\mathbf{w}}\|_{\mathbf{L}^6} + 2\beta \kappa_1 \kappa_2 C_{poi} \|\psi^{\mathbf{w}}\|_{\mathbf{L}^3} \right. \\ &\quad \left. + \kappa_1 \|L\psi^{\mathbf{w}}\|_{\mathbf{L}^2} + 2\mu_r \kappa_2 \|\psi^{\mathbf{w}}\|_{\mathbf{L}^2} \right), \\ \sigma_1 &= \left[ \kappa_8 (\Pi_1 + 1) \|\mathbf{m}\|_{\infty,2} \|\psi^{\mathbf{u}}\|_{\mathbf{L}^\infty} + \beta (\kappa_4 + \kappa_5 \eta + \|\mathbf{m}_t\|_{\infty,2}) \|\psi^{\mathbf{u}}\|_{\mathbf{L}^2} \right] \quad \text{and} \\ \sigma_2 &= \left[ \kappa_8 \|\mathbf{q}\|_{\infty,2} \|\psi^{\mathbf{w}}\|_{\mathbf{L}^\infty} + \beta \|\mathbf{q}_t\|_{\infty,2} \|\psi^{\mathbf{w}}\|_{\mathbf{L}^2} \right]. \end{aligned}$$

Thus, by (5.14), we follow that there exists  $\Theta_i$ ,  $i = 1, 2, 3$  independent of  $\eta$  such that  $\|\mathcal{R}(f, g)\|_{[H^1([0,T])]}^2 \leq \Theta_1 + (\Theta_2 + \Theta_3 \eta)^2$ , which implies the proof of the Lemma.  $\square$

**Lemma 5.2.**  $\mathcal{R}$  is weakly continuous from  $D$  into  $D$ .

*Proof.* It is clear that if  $(f_n, g_n) \rightharpoonup (f, g)$  weakly in  $[H^1(0, T)]^2$ , then  $\mathcal{R}(f_n, g_n) \rightharpoonup \mathcal{R}(f, g)$  weakly in  $[H^1(0, T)]^2$  as well.  $\square$

**Lemma 5.3.** Assume that  $(H_1)$ – $(H_7)$  are satisfied. Then there exists  $k_o$  such that  $\mathcal{R}^{k_o}$  is a contraction map in  $L^2(0, T)$ .

*Proof.* Let us consider  $(f_i, g_i) \in D$  and  $\{u_i, w_i, \rho_i, p_i, h_i\}$ , for  $i = 1, 2$ , the corresponding solution of the direct problem. Then, by the definition of the operator  $\mathcal{R}$  we deduce that there exists  $C > 0$  independent of  $\{u_i, w_i, \rho_i, p_i, h_i, f_i, g_i\}$  such that

$$\begin{aligned} \|\mathcal{R}(f_1, g_1) - \mathcal{R}(f_2, g_2)\|_{L^2(0,t)} &\leq C \left[ \|\rho_1 - \rho_2\|_{L^2(Q_t)} + \|u_1 - u_2\|_{L^2(Q_t)} \right. \\ &\quad \left. + \|w_1 - w_2\|_{L^2(Q_t)} + \|\nabla(h_1 - h_2)\|_{L^2(Q_t)} + \|\nabla(r_1 - r_2)\|_{L^2(Q_t)} \right] \end{aligned}$$

with  $Q_t = \Omega \times [0, t]$ . Hence, by Lemma 4.1 we deduce that

$$\|\mathcal{R}(f_1, g_1) - \mathcal{R}(f_2, g_2)\|_{L^2(0,t)} \leq \left( C \int_0^t \|(f_1, g_1) - (f_2, g_2)\|_{L^2(0,\tau)} d\tau \right)^{1/2}.$$

By induction on  $n$ , we deduce that

$$\|\mathcal{R}^n(f_1, g_1) - \mathcal{R}^n(f_2, g_2)\|_{L^2(0,T)} \leq \left( \frac{C^n T^n}{n!} \right)^{1/2} \|(f_1, g_1) - (f_2, g_2)\|_{L^2(0,T)},$$

which implies the conclusion of the Lemma.  $\square$

**5.3. Proof of Theorem 5.1.** Thanks to Theorem 5.2 we follow that the cornerstone to prove Theorem 5.1 is the proof of the unique solvability of the operator equation (5.6). To this end, we first note that the Lemmas 5.1 and 5.2 guarantee that the operator  $\mathcal{R}$  satisfies the hypothesis of the following Tikhonov fixed point theorem:

**Theorem 5.3.** *Let  $D$  be a non-empty bounded closed convex subset of a separable reflexive Banach space  $E$  and let  $\mathcal{R} : D \rightarrow D$  be a weakly continuous mapping. Then  $\mathcal{R}$  has at least one fixed point in  $D$ .*

Hence, there exists a solution of (5.6). Meanwhile, the local uniqueness follows by Lemma 5.3

#### ACKNOWLEDGMENT

We acknowledge the support of the research projects 124109 3/R (Universidad del Bío-Bío, Chile), 121909 GI/C (Universidad del Bío-Bío, Chile), Fondecyt 1120260 and MTM 2012-32325 (Spain).

#### APPENDIX

##### APPENDIX A. CONSTANTS $\Gamma_i^u$ AND $\Gamma_i^w$ FOR $i \in \{1, 2, 3\}$

We denote the right side of (4.13) by  $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$  where  $\mathcal{E}_i$ ,  $i \in \{1, 2, 3\}$ , are defined and bounded as follows

$$\begin{aligned} \mathcal{E}_1 &:= \|(\mathbf{u}_2)_t(\cdot, t)\|_{L^s} \|\delta\rho(\cdot, t)\|_{L^2} \|\delta\mathbf{u}(\cdot, t)\|_{L^s} + \|\mathbf{u}_1(\cdot, t)\|_{L^\infty} \|\nabla\mathbf{u}_2(\cdot, t)\|_{L^s} \|\delta\rho(\cdot, t)\|_{L^2} \|\delta\mathbf{u}(\cdot, t)\|_{L^s} \\ &\quad + |f_1(t)| \|\nabla h_1(\cdot, t)\|_{L^s} \|\delta\rho(\cdot, t)\|_{L^2} \|\delta\mathbf{u}(\cdot, t)\|_{L^s} + |f_1(t)| \|\mathbf{m}(\cdot, t)\|_{L^\infty} \|\delta\rho(\cdot, t)\|_{L^2} \|\delta\mathbf{u}(\cdot, t)\|_{L^2} \\ &\leq C_{e_1}(t) \|\delta\rho(\cdot, t)\|_{L^2}^2 + \frac{3\epsilon_u}{2} C_{poi} \|\nabla\delta\mathbf{u}(\cdot, t)\|_{L^2}^2 + \frac{\epsilon_u}{2\alpha} \|\sqrt{\delta_1}\delta\mathbf{u}(\cdot, t)\|_{L^2}^2, \\ \mathcal{E}_2 &:= \|\rho_2(\cdot, t)\|_{L^\infty} \|\nabla\mathbf{u}_2(\cdot, t)\|_{L^s} \|\delta\mathbf{u}(\cdot, t)\|_{L^2} \|\delta\mathbf{u}(\cdot, t)\|_{L^s} + 2\mu_r \|\nabla\delta\mathbf{w}(\cdot, t)\|_{L^2} \|\delta\mathbf{u}(\cdot, t)\|_{L^2} \\ &\quad + |f_2(t)| \|\nabla\delta h(\cdot, t)\|_{L^2} \|\rho_2(\cdot, t)\|_{L^\infty} \|\delta\mathbf{u}(\cdot, t)\|_{L^2}, \\ &\leq C_{e_2}(t) \|\sqrt{\delta_1}\delta\mathbf{u}(\cdot, t)\|_{L^2}^2 + \frac{\epsilon_u}{2} C_{poi} \|\nabla\delta\mathbf{u}(\cdot, t)\|_{L^2}^2 + \frac{\epsilon_h}{2} \|\nabla\delta h(\cdot, t)\|_{L^2}^2 + \frac{\epsilon_w}{2} \|\nabla\delta\mathbf{w}(\cdot, t)\|_{L^2}^2, \\ \mathcal{E}_3 &:= |\delta f(t)| \|\nabla h_1(\cdot, t)\|_{L^s} \|\rho_2(\cdot, t)\|_{L^\infty} \|\delta\mathbf{u}(\cdot, t)\|_{L^2} |\Omega|^{1/\bar{s}} \\ &\quad + |\delta f(t)| \|\mathbf{m}(\cdot, t)\|_{L^\infty} \|\rho_2(\cdot, t)\|_{L^\infty} \|\delta\mathbf{u}(\cdot, t)\|_{L^2} |\Omega|^{1/2} \\ &\leq C_{e_3}(t) |\delta f(t)|^2 + \frac{\epsilon_u}{\alpha} \|\sqrt{\rho_1}\delta\mathbf{u}(\cdot, t)\|_{L^2}^2, \end{aligned}$$

where

$$\begin{aligned} C_{e_1}(t) &= \frac{1}{2\epsilon_u} \left\{ \|(\mathbf{u}_2)_t(\cdot, t)\|_{L^s}^2 + |f_1(t)|^2 \|\nabla h_1(\cdot, t)\|_{L^s}^2 \right. \\ &\quad \left. + |f_1(t)|^2 \|\mathbf{m}(\cdot, t)\|_{L^\infty}^2 + \|\mathbf{u}_1(\cdot, t)\|_{L^\infty}^2 \|\nabla\mathbf{u}_2(\cdot, t)\|_{L^s}^2 \right\}, \\ C_{e_2}(t) &= \frac{1}{\alpha} \left\{ \frac{1}{2\epsilon_u} \|\rho_2(\cdot, t)\|_{L^\infty}^2 \|\nabla\mathbf{u}_2(\cdot, t)\|_{L^s}^2 + \frac{2\mathbf{u}_r^2}{\epsilon_w} + \frac{1}{\epsilon_h} |f_2(t)|^2 \|\rho_2(\cdot, t)\|_{L^\infty}^2 \right\} \text{ and} \\ C_{e_3}(t) &= \frac{1}{2\epsilon_u} \left\{ \|\nabla h_1(\cdot, t)\|_{L^s}^2 |\Omega|^{2/\bar{s}} + \|\mathbf{m}(\cdot, t)\|_{L^\infty}^2 |\Omega| \right\} \|\rho_2(\cdot, t)\|_{L^\infty}^2. \end{aligned}$$

Then, we deduce the inequality (4.14) with

$$\Gamma_1^u(t) = \max \left\{ C_{e_2}(t) + \frac{3\epsilon_u}{2\alpha}, C_{e_1}(t) \right\}, \quad \Gamma_2^u(t) = \frac{\epsilon_h}{2} \quad \text{and} \quad \Gamma_3^u(t) = C_{e_3}(t).$$

Meanwhile, for (4.16) we proceed in a similar way and obtain that

$$\Gamma_1^w(t) = \max \left\{ C_{f_2}(t) + \frac{3\epsilon_w}{2\alpha}, C_{f_1}(t) \right\}, \quad \Gamma_2^w(t) = \frac{\epsilon_g}{2} \quad \text{and} \quad \Gamma_3^w(t) = C_{f_3}(t),$$

with

$$C_{f_1}(t) = \frac{1}{2\epsilon_w} \left\{ \|(\mathbf{w}_2)_t(\cdot, t)\|_{L^s}^2 + |g_1(t)|^2 \|\nabla r_1(\cdot, t)\|_{L^s}^2 \right\}$$

$$\begin{aligned}
& + |g_1(t)|^2 \|\mathbf{q}(\cdot, t)\|_{L^\infty}^2 + \|\mathbf{w}_1(\cdot, t)\|_{L^\infty}^2 \|\nabla \mathbf{w}_2(\cdot, t)\|_{L^s}^2 \Big\}, \\
C_{f_2}(t) &= \frac{1}{\alpha} \left\{ \frac{1}{2\epsilon_{\mathbf{w}}} \|\rho_2(\cdot, t)\|_{L^\infty}^2 \|\nabla \mathbf{w}_2(\cdot, t)\|_{L^s}^2 + \frac{2\mathbf{u}_r^2}{\epsilon_{\mathbf{u}}} + \frac{1}{\epsilon_g} |g_2(t)|^2 \|\rho_2(\cdot, t)\|_{L^\infty}^2 \right\} \text{ and} \\
C_{f_3}(t) &= \frac{1}{2\epsilon_{\mathbf{w}}} \left\{ \|\nabla r_1(\cdot, t)\|_{L^s}^2 |\Omega|^{2/s} + \|\mathbf{q}(\cdot, t)\|_{L^\infty}^2 |\Omega| \right\} \|\rho_2(\cdot, t)\|_{L^\infty}^2.
\end{aligned}$$

## REFERENCES

## REFERENCES

- [1] E.L. Aero, A.N. Bulygin, and E.V. Kuvshinskii. Asymmetric hydromechanics. *J. Appl. Math. Mech.* 29(2):333–346, 1965.
- [2] G. Anger. *Inverse Problems in Differential Equations*. Plenum Press, New York, 1990.
- [3] S.N. Antontsev, A.V. Kazhikhov, and V.N. Monakhov. *Boundary value problems in mechanics of nonhomogeneous fluids*, Studies in Mathematics and its Applications, Vol. 22, North-Holland Publishing Co., Amsterdam, 1990.
- [4] M. Ashraf, M. Anwar K., and K.S. Syed. Numerical study of asymmetric laminar flow of micropolar fluids in a porous channel. *Computers & Fluids*, 38(10):1895–1902, 2009.
- [5] J.L. Boldrini and M. Rojas-Medar. On the convergence rate of spectral approximation for the equations for nonhomogeneous asymmetric fluids. *RAIRO Modél. Math. Anal. Numér.*, 30(2):123–155, 1996.
- [6] J.L. Boldrini, M.A. Rojas-Medar, and E. Fernández-Cara. Semi-Galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids, *J. Math. Pures Appl. (9)*, 82(11):1499–1525, 2003.
- [7] P. Braz e Silva, F.W. Cruz, and M. Rojas-Medar. Vanishing viscosity for non-homogeneous asymmetric fluids in  $\mathbb{R}^3$ : the  $L^2$  case. *J. Math. Anal. Appl.* 420(1):207–221, 2014.
- [8] P. Braz e Silva, E. Fernández-Cara, M.A. Rojas-Medar, Vanishing viscosity for non-homogeneous asymmetric fluids in  $\mathbb{R}^3$ . *J. Math. Anal. Appl.* 332(2):833–845, 2007.
- [9] M. Choulli, O.Y. Imanuvilov, J.-P. Puel, and M. Yamamoto. Inverse source problem for linearized NavierStokes equations with data in arbitrary sub-domain. *Applicable Analysis: An International Journal*, 92(10):2127–2143, 2013.
- [10] C. Conca, R. Gormaz, E.E. Ortega-Torres, and M.A. Rojas-Medar, The equations of non-homogeneous asymmetric fluids: an iterative approach. *Math. Methods Appl. Sci.* 25(15):1251–1280, 2002.
- [11] L. C. Evans. *Partial Differential Equations*(Second edition), Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, RI, 2010.
- [12] A.C. Eringen. Simple microfluids. *Int. J. Eng. Sci.*, 2(2):205–217, 1964.
- [13] A.C. Eringen. Theory of micropolar fluids. *J. Math. Mech.* 16(1):1–16, 1966.
- [14] J. Fan and G. Nakamura. Local solvability of an inverse problem to the density-dependent Navier-Stokes equations. *Appl. Anal.*, 87(10-11):1255–1265, 2008.
- [15] E. Fernández-Cara, T. Horsin, and H. Kasumba. Some inverse and control problems for fluids. *Annales mathématiques Blaise Pascal*, 20:101–138, 2013.
- [16] J.G. Heywood. The Navier-Stokes equations: on the existence, regularity and decay of solutions. *Indiana Univ. Math. J.* 29(5):639–681, 1980.
- [17] V. Isakov. *Inverse source problems*. (Mathematical Surveys and Monographs, Volume 34), American Mathematical Society, Providence, RI, 1990.
- [18] V. Isakov. *Inverse Problems for PDE*, Springer-Verlag, New York, 2006.
- [19] Y.C. Kim. Yang-Yang anomalies and coexistence diameters: simulation of asymmetric fluids. *Phys. Rev. E*, 71(5):051501, 15 pp., 2005.
- [20] O. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, New York, 1969.
- [21] P. L. Lions. *Mathematical Topics in Fluid Mechanics, Incompressible Models* (Volume 1), Claredon Press, Oxford Science Publications, Oxford, 1996.
- [22] P. L. Lions. *Mathematical Topics in Fluid Mechanics, Compressible Models* (Volume 2), Claredon Press, Oxford Science Publications, Oxford, 1998.
- [23] Łukaszewicz G. *Micropolar Fluids: Theory and Applications*. Birkhauser, Basel, 1999.
- [24] G. Łukaszewicz. On nonstationary flows of incompressible asymmetric fluids. *Math. Methods Appl. Sci.* 13(3):219–232, 1990.
- [25] G. Łukaszewicz, Grzegorz, and W. Waluś. On stationary flows of asymmetric fluids with heat convection. *Math. Methods Appl. Sci.* 11(3):343–351, 1989.
- [26] A.I. Prilepko, D.G. Orlovsky, and I.A. Vasin. *Methods for solving inverse problems in mathematical physics*. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 231. Marcel Dekker Inc., New York, 2000.
- [27] L. Petrosyan. *Some Problems of Mechanics of Fluids with Antisymmetric Stress Tensor*(in Russian). Erevan, 1984.
- [28] M. A. Rojas-Medar, and E.E. Ortega-Torres, The equations of a viscous asymmetric fluid: an interactive approach. *ZAMM Z. Angew. Math. Mech.* 85(7):471–489, 2005.

- [29] J. Simon. Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure. *SIAM J. Math. Anal.*, 21(5):1093–1117, 1990.
- [30] R. Temam. *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2, North-Holland Publishing Co., Amsterdam, 1977.
- [31] I.A. Vasin. The existence and uniqueness of the generalized solution of the inverse problem for the nonlinear nonstationary Navier-Stokes system in the case of integral overdetermination. *Mathematical Notes*, 54(4):1002–1009, 1993
- [32] F. Vitoriano e Silva. On the steady viscous flow of a nonhomogeneous asymmetric fluid. *Ann. Mat. Pura Appl.* 192(4):665–672, 2013.